Section 3.7: 2, 5 (optional)

2. Let (X, d) be a complete metric space, and let $f : X \to X$ be a mapping such that

$$f^{(p)} \doteq \underbrace{f \circ f \circ \cdots \circ f}_{p}$$

is a contraction for some integer $p \geq 2$, with contraction constant $\gamma \in [0, 1)$. In other words, $d(f^{(p)}(x), f^{(p)}(y)) \leq \gamma d(x, y)$ for all $x, y \in X$. Note that the fact that $f^{(p)}$ is a contraction does not imply that f is continuous. Show that there is one and only one fixed point x of f, and that, for any $x_0 \in X$, the sequence generated by $x_{n+1} = f(x_n)$ for $n \geq 0$ converges to x.

5. Let (X, d) be a metric space, $A \subset X$, and $L \geq 0$. Suppose that $f : A \to \mathbb{R}$ is Lipschitz continuous (on A), with Lipschitz constant L. Show that there is a $g : X \to \mathbb{R}$ such that $g(x) = f(x)$ for all $x \in A$, and that g is Lipschitz continuous (on X), with Lipschitz constant L. This result is often called the MacShane Lemma.