
Section 3.1: A, 2

- A. Let (X, d) be a metric space and $A \subset X$ be a non-empty subset. Recall that $d(x, A) \doteq \inf\{d(x, y) : y \in A\}$. Theorem 11.1.3 asserts that

$$|d(x, A) - d(y, A)| \leq d(x, y) \text{ for all } x, y \in A .$$

Prove this result.

2. Suppose that $(X, \|\cdot\|)$ is a uniformly convex Banach space, and let $Z \subset X$ be a non-empty, closed, convex subset. Let $P : X \rightarrow Z$ be the projection operator defined in Theorem 3.1.7. Show that P is continuous. **Hint:** Let $(x_n)_{n \in \mathbb{N}}$ be a sequence converging to $x \in X$. Argue that $\limsup \|x - P(x_n)\| \leq \|x - P(x)\|$ and $\liminf \|x - P(x_n)\| \geq \|x - P(x)\|$, and deduce from this that $\lim \|x - P(x_n)\| = \|x - P(x)\|$. You may then “recycle” (or cite) arguments from the proof of Theorem 3.1.7 to ultimately argue that $\lim \|P(x_n) - P(x)\| = 0$.