
Section 2.5: 1, 2 (optional), 3 (optional), 4

1. (a) Given any $x \in \mathbb{K}^n$, show that $\|x\|_\infty = \lim_{p \rightarrow \infty} \|x\|_p$.
- (b) Given any $x \in \ell^\infty$, show that

$$\|x\|_\infty = \lim_{n \rightarrow \infty} \left(\lim_{p \rightarrow \infty} \left(\sum_{k=1}^n |x_k|^p \right)^{1/p} \right)$$

Hint: For the first part, argue that $\|x\|_\infty \leq \|x\|_p \leq n^{1/p} \|x\|_\infty$.

2. **(Optional)** The exercise concerns the conditions under which equality is achieved in Hölder's Inequality and Minkowski's Inequality.

- (a) Let $p, q > 1$ be conjugate exponents. Show that equality in Hölder's Inequality,

$$\sum_{k=1}^{\infty} |x_k y_k| \leq \|x\|_p \|y\|_q \text{ for } x \in \ell^p, y \in \ell^q, \quad (1)$$

holds if and only if there are $a, b \geq 0$, at least one of which is positive, such that $a|x_k|^p = b|y_k|^q$ for all $k \in \mathbb{N}$.

- (b) Let $p > 1$. Show that equality holds in Minkowski's Inequality

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p \text{ for } x, y \in \ell^p, \quad (2)$$

if and only if there are $a, b \geq 0$, at least one of which is positive, for which $ax_k = by_k$ for all $k \in \mathbb{N}$.

3. **(Optional)** Let $0 < p < 1$, and $X = \{x \in \mathbb{K}^{\mathbb{N}} : \sum_{k=1}^{\infty} |x_k|^p < \infty\}$.
 - (a) Show that X is a vector space.
 - (b) Show that the mapping $x \mapsto \rho(x) \doteq (\sum_{k=1}^{\infty} |x_k|^p)^{1/p}$ is not a norm on X .
 - (c) Define $d : X \times X \rightarrow \mathbb{R}$ by $d(x, y) = \sum_{k=1}^{\infty} |x_k - y_k|^p$ for $x, y \in X$. Show that d is a metric on X .
4. Let p and q satisfy $0 < p < q < \infty$ (do not assume $p \geq 1$), and let $x \in \mathbb{K}^{\mathbb{N}}$ satisfy $\sum_{k=1}^{\infty} |x_k|^p < \infty$. Show that the series $\sum_{k=1}^{\infty} |x_k|^q$ converges, and that Jensen's inequality holds:

$$\left(\sum_{k=1}^{\infty} |x_k|^q \right)^{1/q} \leq \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{1/p}. \quad (3)$$

Note: There are a variety of related and important inequalities that go by the name “Jensen’s inequality”. Although you might find a version that can be properly applied to yield (3), it is probably simpler to just prove (3) directly.

Hint: The case $x = 0$ is trivial, so note this in your argument, and quickly move onto the case $x \neq 0$. There will be an N such that, for $n \geq N$, $\sum_{k=1}^n |x_k|^q > 0$. Fix $n \geq N$, and consider $\sum_{k=1}^n |y_k|^q$ and $\sum_{k=1}^n |y_k|^p$, where $y_k = |x_k|/(\sum_{k=1}^n |x_k|^q)^{1/q}$. How do these two sums compare with each other?