
Section 2.4: 1, 2 (optional)

1. (Dini's Theorem) Let K be a compact metric space, let $(f_n)_{n=1}^\infty$ be a sequence in $C(K)$ and $f \in C(K)$. Suppose that $f_n \rightarrow f$ pointwise, and that $f_n(x) \leq f_m(x)$ for all $x \in K$ when $n \leq m$. Show that $f_n \rightarrow f$ uniformly.

Hint: Fix $\varepsilon > 0$ and define $\mathcal{O}_n = \{x \in K : f(x) - f_n(x) < \varepsilon\}$. Argue that these sets are open and nested, $\mathcal{O}_n \subset \mathcal{O}_{n+1}$, for all $n \in \mathbb{N}$. Argue that pointwise convergence implies that $\{\mathcal{O}_n : n \in \mathbb{N}\}$ is an open cover of K , and use the compactness of K accordingly.

2. **(Optional)** Let $\Omega \subset \mathbb{R}^n$ be a non-empty open subset. For $f \in C(\Omega)$ and compact subsets $K \subset \Omega$, we define

$$|f|_K \doteq \sup_{x \in K} |f(x)| .$$

For each such K , $|\cdot|_K$ is a seminorm on $C(\Omega)$; strict positivity fails because there will always be non-zero continuous functions that vanish indentially on K .

- (a) Let $(K_i)_{i=1}^\infty$ be a sequence of subsets of Ω such that

$$K_i \subset \text{int}(K_{i+1}) \text{ for all } i \geq 1 \text{ and } \bigcup_{i=1}^\infty K_i = \Omega .$$

Such a sequence of nested sets is guaranteed by Theorem 1.13.6. Let $(\alpha_i)_{i=1}^\infty$ be a sequence of positive numbers such that $\sum_{i=1}^\infty \alpha_i$ converges. Now define $d : C(\Omega) \times C(\Omega) \rightarrow \mathbb{R}$ by

$$d(f, g) \doteq \sum_{i=1}^\infty \frac{\alpha_i |f - g|_{K_i}}{1 + |f - g|_{K_i}} .$$

Show that d is a metric on $C(\Omega)$.

- (b) Show that a sequence $(f_m)_{m=1}^\infty$ in $C(\Omega)$ converges to f in the metric space $(C(\Omega), d)$ if and only if $|f_m - f|_K \rightarrow 0$ as $j \rightarrow \infty$ for each K .
- (c) Show that the metric vector space $(C(\Omega), d)$ is complete.