
Section 2.2:1 (optional), 3 (optional), 4, A, B

1. Let $X = \mathbb{R}^{n \times n}$ be the space of all $n \times n$ matrices with real entries, equipped with the Frobenius norm, $\|A\| = \left(\sum_{i,j=1}^n |a_{ij}|^2 \right)^{1/2}$.

- (a) Show that the set of all invertible matrices in $\mathbb{R}^{n \times n}$ (i.e. the general linear group $GL_n(\mathbb{R})$) is open.
- (b) Show that the set of all symmetric matrices is closed.

Hint: You may use properties of the Frobenius norm such as $\|AB\| \leq \|A\| \|B\|$ and $\|A\mathbf{x}\|_2 \leq \|A\| \|\mathbf{x}\|_2$, where $\|\mathbf{y}\|_2 = (|y_1|^2 + \cdots + |y_n|^2)^{1/2}$ is the Euclidean norm for $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$. For the first problem, aim to show that $B(A, 1/\|A^{-1}\|) \subset GL_n(\mathbb{R})$ when $A \in GL_n(\mathbb{R})$.

3. Let $\|\cdot\|$ and $\|\!\|\cdot\!\|$ be two norms on X , and let $(x_n)_{n=1}^\infty$ be a sequence in X . Suppose that $\|x_n - x\| \rightarrow 0$ for some $x \in X$ and $\|\!\|x_n - y\!\| \rightarrow 0$ for some $y \in X$. Must it be the case that $y = x$?

Hint: If $\|\cdot\|$ and $\|\!\|\cdot\!\|$ are equivalent norms, then $y = x$, so if you are looking for counter-examples, the norms cannot be equivalent, which implies that you must consider infinite dimensional vector spaces. One possibility is to consider the space $\mathcal{P}$ of polynomials of arbitrary (but finite) order, and construct norms in terms of coefficients with respect to different bases.

4. Let K be a (non-empty) compact subset of the normed vector space $(X, \|\cdot\|)$.
- (a) Show that, for each $x \in X$ there is a $y \in K$ such that $\|x - y\| = \inf_{z \in K} \|x - z\|$.
- (b) Show that, if for each $x \in X$ there is a unique(!) $y \in K$ such that $\|x - y\| = \inf_{z \in K} \|x - z\|$, then the mapping defined by $P(x) = y$, is continuous.

Note: You should not assume that P is a linear mapping.

Hint: For the first part, you should first argue that, for a fixed x , the function $f : X \rightarrow \mathbb{R}$ defined by $f(z) = \|x - z\|$ is continuous. For the second part, you should first argue that the function $d_K : X \rightarrow \mathbb{R}$ defined by $d_K(x) = \inf_{z \in K} \|x - z\|$ is continuous.

- A. Suppose that Z is a subspace of the vector space X and that $\dim(X/Z) = 1$. Show that, for each $y \in X \setminus Z$, we have $X = Z \oplus \text{span}\{y\}$. Note: When Z is a subspace of X , $\dim(X/Z)$ is typically called the *codimension* of Z in X .

- B. (Complexifying a real normed vector space). Let $(X, \|\cdot\|)$ be a normed vector space over $\mathbb{R}$. We define the complexification of X by

$$X_{\mathbb{C}} = \{x + iy : x, y \in X\} = X \oplus iX ,$$

together with the operations

$$(VA) \quad (x + iy) + (u + iv) \doteq (x + u) + i(y + v) \qquad x, y, u, v \in X$$

$$(SM) \quad (a + ib)(x + iy) \doteq (ax - by) + i(ay + bx) \qquad a, b \in \mathbb{R}, x, y \in X$$

This defines a complex vector space—you can verify the vector space axioms if you wish, but I am not requesting it. The most common (and natural) extension of the norm $\|\cdot\|$ to $X_{\mathbb{C}}$ is as follows:

$$\|x + iy\|_{\mathbb{C}} \doteq \sup_{\theta \in \mathbb{R}} \|(\cos \theta)x + (\sin \theta)y\|$$

- (a) Show that $\|\cdot\|_{\mathbb{C}} = \|\cdot\|$ on X . (This is not meant to be tricky).
- (b) Show that $\|\cdot\|_{\mathbb{C}}$ satisfies the properties of a norm on $X_{\mathbb{C}}$. Note: Homogeneity is the trickiest property to establish.