
Section 2.12: A, B, 6; Optional: C, 1, 5

A. Let $A : X \rightarrow Y$ be a linear operator between these vector spaces.

- (a) Show that the kernel $\text{Ker}(A)$ and direct image $\text{Im}(A)$ are subspaces of their respective vector spaces.
- (b) Show that, if X and Y are normed vector spaces and A is continuous, then $\text{Ker}(A)$ is closed.
- (c) Let $X = C^1[0, 1]$ with the sup-norm $\|f\| = \sup_{x \in [0, 1]} |f(x)|$, and let $Y = \mathbb{R}$ with the absolute value norm. Let $A : X \rightarrow Y$ be given by $Af \doteq f'(1)$. Show that A is a linear operator but that $\text{Ker}(A)$ is not closed. Hint: Consider the sequence of polynomials $p_n(x) = x - x^n/n$ for $n \in \mathbb{N}$.

B. Let X be a vector space over $\mathbb{K}$ and let $\phi : X \rightarrow \mathbb{K}$ be a linear functional.

- (a) Let U be a subspace of X . Show that if $\text{Ker}(\phi) \subset U$, then $\text{Ker}(\phi) = U$ or $U = X$.
- (b) Let $\psi : X \rightarrow \mathbb{K}$ also be a linear functional. If $\text{Ker}(\phi) \subset \text{Ker}(\psi)$, then there is a $\beta \in \mathbb{K}$ such that $\psi = \beta\phi$.
- (c) If $\text{Ker}(\phi) \neq X$ and $z \in X \setminus \text{Ker}(\phi)$, then $X = \text{Ker}(\phi) \oplus \text{span}\{z\}$.

Hint: You will get a lot of mileage out of constructions of the form $x = x - \alpha z + \alpha z$, where $\alpha = \phi(x)/\phi(z)$.

C. **(Optional)** Show that the space $\mathcal{L}(X, Y)$ of bounded linear operators from X to Y is a vector space, under the usual definitions of operator/function addition and scalar multiplication. Don't verify all of the properties of that must hold for a vector space, but only show that $\mathcal{L}(X, Y)$ is closed under addition and scalar multiplication.

1. **(Optional)** Let $\|\cdot\|_p$ for $1 \leq p \leq \infty$ be the norms on $\mathbb{K}^n$ defined by in Theorem 2.2.2, and identify a matrix $A \in \mathbb{K}^{n \times n}$ with a linear operator $F \in \mathcal{L}(\mathbb{K}^n)$ in the usual way, i.e.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \leftrightarrow F(x) = Ax = \begin{pmatrix} \sum_{j=1}^n a_{1j}x_j \\ \sum_{j=1}^n a_{2j}x_j \\ \vdots \\ \sum_{j=1}^n a_{nj}x_j \end{pmatrix}$$

(a) Show that

$$\begin{aligned}\|A\|_1 &\doteq \sup_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_1} = \max_{1 \leq j \leq n} \sum_{i=1}^n |a_{ij}| , \\ \|A\|_\infty &\doteq \sup_{x \neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} = \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}| , \\ \|A\|_2 &\doteq \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \max\{|\lambda|^{1/2} : \lambda \text{ is an eigenvalue of } A^*A\} ,\end{aligned}$$

where A^* is the conjugate transpose of A when A is complex, and just the transpose of A when A is real.

(b) Find formulas in the spirit of part (a) for the operator norms defined by

$$\|A\|_{p,1} \doteq \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_1} \quad (1 < p \leq \infty) \quad , \quad \|A\|_{\infty,p} \doteq \sup_{x \neq 0} \frac{\|Ax\|_\infty}{\|x\|_p} \quad (1 \leq p < \infty) .$$

5. **(Optional)** Show that a normed vector space $(X, \|\cdot\|)$ is finite dimensional if and only if all linear functionals on X are continuous.
6. Let X and Y be two normed vector spaces over the same field $\mathbb{K}$, and let $A : X \rightarrow Y$ be a linear operator.
 - (a) We define $[A] : X/\text{Ker } A \rightarrow \text{Im } A$ by $[A][x] \doteq Ax$ for each $[x] \in X/\text{Ker } A$. Show that this definition makes sense. At issue here is the fact that, if $\tilde{x} \in [x]$, then $[x] = [\tilde{x}]$, so we need $[A][\tilde{x}] = [A][x]$ if the definition is to make sense.
 - (b) Show that, if A is continuous, then $[A]$ is continuous and $\|[A]\| = \|A\|$.