
Section 2.10: 1, 2 (optional), A (optional)

1. Let Y be a finite dimensional subspace of a normed vector space $(X, \|\cdot\|)$.
 - (a) Show that, for any $x \in X$, there is a $\tilde{y} \in Y$ (not necessarily unique) such that $\|x - \tilde{y}\| = \inf_{y \in Y} \|x - y\|$.
 - (b) Suppose that $(X, \|\cdot\|)$ is *strictly convex*, i.e. whenever $u, v \in X$ are distinct and satisfy $\|u\| = \|v\| = 1$, it follows that $\|u/2 + v/2\| < (1/2) + (1/2) = 1$. Show that the vector $\tilde{y} \in Y$ found above is necessarily unique in this case.
 - (c) Let $n \in \mathbb{N}$, $n \geq 2$. Show that $(\mathbb{K}^n, \|\cdot\|_p)$ is strictly convex for any $p \in (1, \infty)$, but not for $p = 1$ or $p = \infty$.
2. **(Optional)** Show that the interior of a compact subset of an infinite dimensional normed vector space must be empty.
- A. **(Optional)** Let $(X, \|\cdot\|)$ be a finite dimensional normed vector space. Show that any bounded sequence in $(X, \|\cdot\|)$ must have a convergent subsequence.