Landscape Refinement for hp-Adaptive Eigenvalue Approximations

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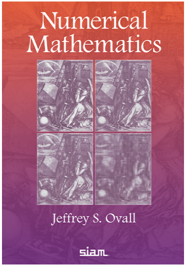


DMS 2136228, *2208056*

Source/Eigenvalue Problem

Source Problem

Find u such that

$$\begin{aligned} \mathcal{L}u &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned}$$

Eigenvalue Problem

Find λ, $\psi \neq 0$ such that

$$\begin{aligned} \mathcal{L}\psi &= \lambda\psi \text{ in } \Omega \\ \psi &= 0 \text{ on } \partial\Omega \end{aligned}$$

Computational Objective

Adaptively approx. (large) collection of eigenpairs $\Lambda = \{(\lambda_j, \psi_j) : j \in J\}$

Collection may contain

- repeated and/or tightly-clustered eigenvalues
- eigenvectors with very different features

A Sampling of Associated Articles

Adaptive Refinement Template

$$\cdots \textit{SOLVE} \longrightarrow \textit{ESTIMATE} \longrightarrow \textit{MARK} \longrightarrow \textit{REFINE} \cdots$$

Five Independent Strands

1. Bank, Giani, Grubišić, Hakula, Międlar, Ovall
   - GrO (MC 2009), GGMO (NM 2016), GGHO (JSC 2021)
2. Boffi, Bonito, Demlow, Gallistl, Gardini, Gastaldi
   - Gal (NM 2015), BonD (SINUM 2016), BofGGG (MC 2017)
3. Cancès, Dusson, Maday, Stamm, Vohralík
   - CDMSV (NM 2018), CDMSV (MC 2020)
4. Liu, Vejchodský
   - LV (NM 2022)
5. Bi, Wang, Yang
   - BWY (SINUM 2024)

Operator and "Eigenfacts"

Differential Operator

$\Omega \subset \mathbb{R}^d$ open, bounded

$$\mathcal{L}v \doteq -\nabla \cdot (A\nabla v) + cv \ , \ \mathrm{Dom}(\mathcal{L}) = \{v \in H_0^1(\Omega) : \ \mathcal{L}v \in L^2(\Omega)\}$$

Eigenfacts

1. $\mathrm{Spec}(\mathcal{L}) = \mathrm{Eig}(\mathcal{L}) = (\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{R}$
   - $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots, \lambda_n \to \infty$
2. Hilbert basis of eigenvectors $(\psi_n)_{n\in\mathbb{N}}$, $\mathcal{L}\psi_n = \lambda_n\psi_n$
   - $(\psi_m, \psi_n) = \delta_{mn}$ and $v = \sum_{n=1}^{\infty}(v, \psi_n)\,\psi_n$ for any $v \in L^2(\Omega)$, where

     $$(f, g) \doteq \int_\Omega f\,\overline{g}\,dx$$

Standard Approach, An Indecent Proposal

Standard Approach

Finite element space should be well-suited to approximating **entire** invariant subspace for Λ, $E(\Lambda)$, not just a basis of it

- Current computed basis used in some (sophisticated) way to estimate error and mark elements and determine how to refine
- Becomes increasingly costly as $\dim E(\Lambda)$ grows

An Indecent Proposal (`arXiv 2501.05311`)

Use computed eigpairs to estimate their error (stopping criterion), but a single source problem for driving adaptivity

$$\textit{ESTIMATE } \mathcal{L}\psi = \lambda\psi \qquad || \qquad \textit{MARK, REFINE } \mathcal{L}u = 1$$

- Need not compute eigenpairs in every adaptive loop
- Refinement based on a single source less costly, cumbersome

Why Might This Not Be Ridiculous?

Fourier Expansion of $u = u(f)$

$$u = \sum_{n\in\mathbb{N}} \frac{(f,\psi_n)}{\lambda_n}\,\psi_n$$

- u "encodes" information about all eigenvectors such that $(f,\psi_n) \neq 0$
- u should be a faithful indicator of where eigenvectors are singular or smooth

Some (Obvious) Caveats, Questions ($\mathcal{L}u = 1$)

- u does not "see" eigenvectors higher in the spectrum as well.
  Singularities higher in the spectrum are already present lower in the spectrum, so u sees them. hp-adaptivity addresses issue of (smooth) oscillatory behavior
- What if u misses the worst kinds of singularities that can be present in the collection? Unlikely (though not impossible). The worst singularities that can occur, typically occur in ψ_1, and $(\psi_1, 1) \neq 0$

Why Drive the Process Using $f = 1$ ($\mathcal{L}u = 1$)?

Theorem (Pointwise Stability, GOP-S, 2025)

If $v \in C(\overline{\Omega})$ and $\mathcal{L}v \in L^\infty(\Omega)$ then

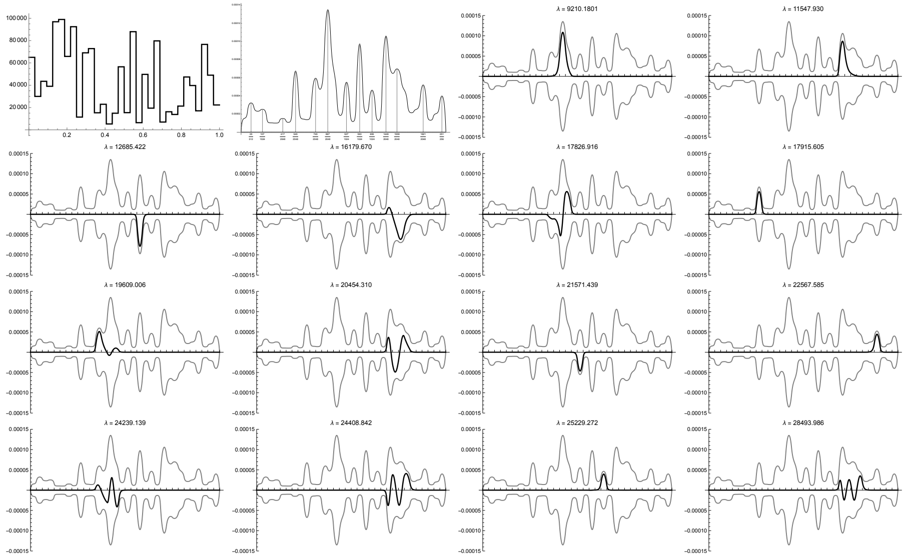
$$|v(x)| \leq \|v\|_{L^\infty(\partial\Omega)} + \|\mathcal{L}v\|_{L^\infty(\Omega)} u(x) \text{ for all } x \in \overline{\Omega}\,,$$

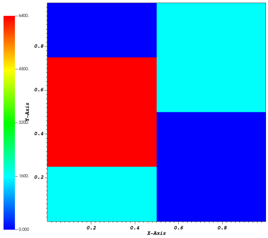
with equality for $v = u$. Consequently, if (λ, ψ) is an eigenpair of $\mathcal{L}$, then

$$\frac{|\psi(x)|}{\lambda \|\psi\|_{L^\infty(\Omega)}} \leq u(x) \text{ for all } x \in \overline{\Omega}\,.$$

1. Landscape function $\mathcal{L}u = 1$ encodes an impressive amount of information about eigenvectors—not just singular/smooth info
   D. Arnold, G. David, D. Jerison, M. Filoche, S. Mayboroda (SISC 2019)
2. $f = 1$ does not introduce any special “biases” to u
   - Occasionally a different source f, or a small(!) collection of sources can be useful

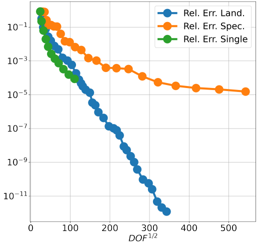
Landscape (Also) Sees Localization, $\mathcal{L} = -\Delta + c$

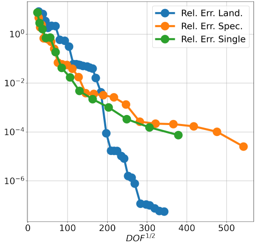


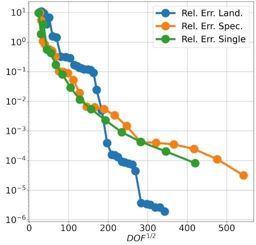


Relative eigenvalue error for single eigenvalues, $\lambda_1, \lambda_{57}, \lambda_{76}$, under three different refinement strategies:

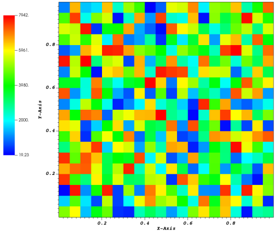
- Current approx. of eigenpair
- Current approx. of first 100 eigenpairs
- Current approx. of landscape function





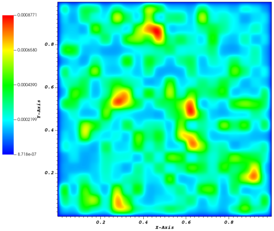


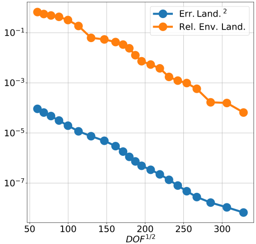
Example 2 $\mathcal{L} = -\Delta + c$

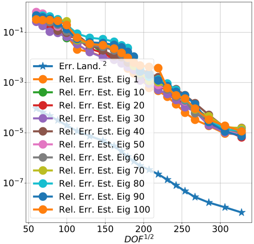


Relative eigenvalue error for several single eigenvalues and a collective measure of error, under landscape refinement, for first $M = 100$ eigenpairs

$$\frac{\eta^2_{\mathrm{eig},j}}{\lambda_{j,h}} \approx \frac{|\lambda_{j,h} - \lambda_j|}{\lambda_{j,h}} \quad , \quad \eta^2_{\mathrm{max,rel}} = \max_{1 \le j \le M} \frac{\eta^2_{\mathrm{eig},j}}{\lambda_{j,h}}$$

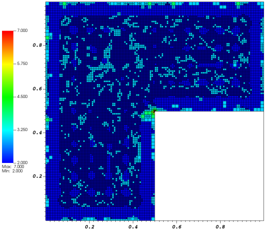


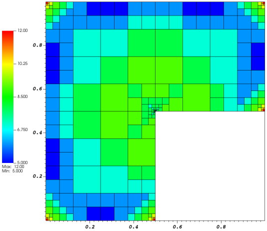


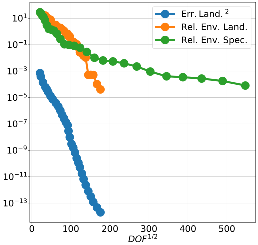


Example 3 $\mathcal{L} = -\Delta$

Collective measure of error $\eta^2_{\max,\text{rel}}$ for first $M = 100$ eigenpairs under cluster refinement and landscape refinement

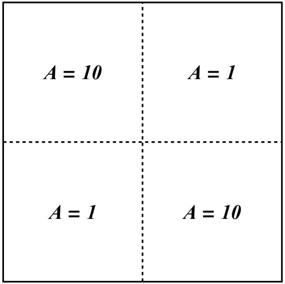




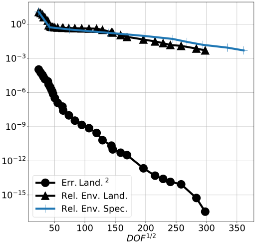


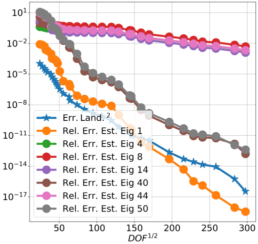
Too many cooks spoil the broth!

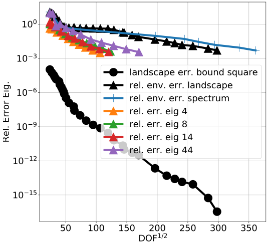
Example 4 $\mathcal{L}v = -\nabla \cdot (A\nabla v)$



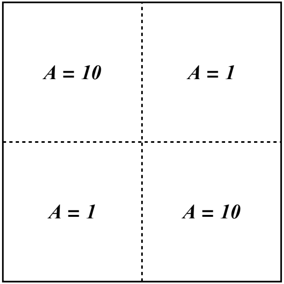
1. $\eta^2_{\max,\mathrm{rel}}$ for first $M = 50$ eigenvalues under landscape and cluster refinement refinement
2. $\frac{\eta^2_{\mathrm{eig},j}}{\lambda_{j,h}}$ for several j under landscape refinment
3. $\frac{\eta^2_{\mathrm{eig},j}}{\lambda_{j,h}}$ for some “bad” j under single eigenvector refinement

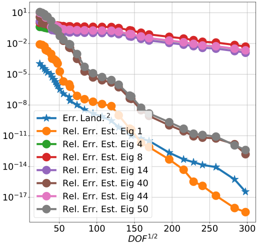


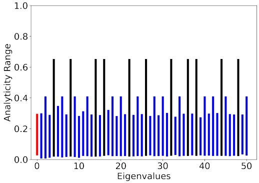




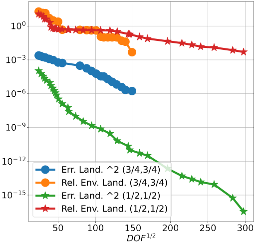
Example 4 What’s Going Wrong Here?

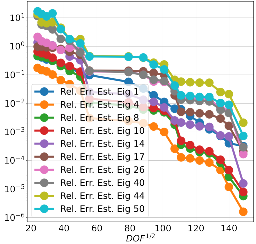


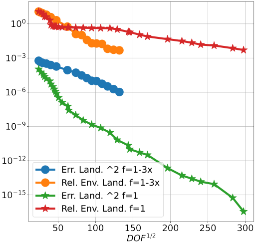




- Empirical measures of analyticity (right) — larger means more singular
- Is it possible that $f = 1$ is orthogonal to all eigenvectors having the strongest singularity (r^α, $\alpha \approx 0.389964$)? Yes!
- Analytic computations on a related problem on the unit disk reveal that this does happen there. Culprit: symmetry in the discontinuity of A

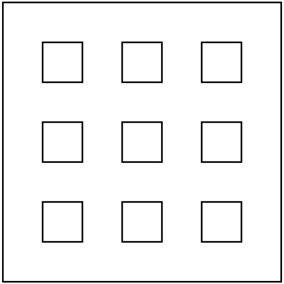


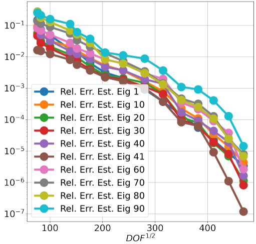


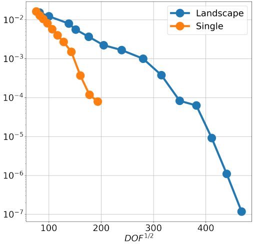


1. Collective measures of error under landscape refinement when singular point is $(1/2, 1/2)$ or $(3/4, 3/4)$
2. Individual relative errors under landscape refinement when singular point is $(3/4, 3/4)$
3. Collective measures of error under refinement when singular point is $(1/2, 1/2)$ and $f = 1$ or $f = 1 - 3x$

Example 5 Suboptimal Convergence of Smooth Eigenvectors







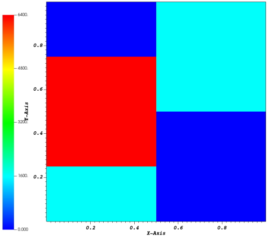
- 36 $r^{2/3}$-type singularities in u, and same singularity at at least one corner in several eigenvectors among first $M = 100$
- $\psi_{41} = 2\sin(7\pi x)\sin(7\pi y)$ is analytic (only one in first M)
- Landscape refinement drives down $|\lambda_{41,h} - \lambda_{41}|/\lambda_{41}$ more slowly that does single eigenvector refinement — no surprise
- Landscape approach designed for large(r) collections of eigenpairs and collective error. If singularities can occur in eigenvectors, such collections will likely include one.

Final Remarks

- Landscape refinement provides an attractive alternative to standard adaptive approaches
  - Error estimation for termination of adaptive loop decoupled from marking strategy
  - Error estimation, or even eigenpairs(!), not necessary for every loop (certainly early on), and landscape refinement is simple
- Even though an estimate of the form $\|\psi_j - \psi_{j,h}\|_{H^1(\Omega)} \leq C\|u - u_h\|_{H^1(\Omega)}$ may appear to hold (and may actually hold) in practice for many examples, such a result cannot hold in general—but it doesn't need to hold for the refinement to be doing a good job
- Might instead use landscape refinement to "shepherd" process along until FE space sufficiently rich to begin to resolve features of interest, and then shift to a different approach
- Still in need of further (numerical) analysis

Pausing the Eigensolver in Example 1

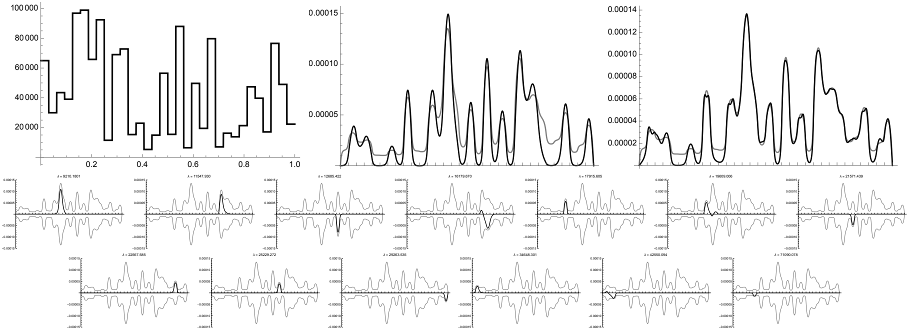
Pause Length	N Meshes	DOF	CPU Time	Reduction(%)
0	36	117805	30986.7	0
1	37	128366	19291.4	37.7428
2	37	128366	14276	53.9284
3	37	128366	11672.5	62.3306
4	36	117805	8940.47	71.1473
5	37	128366	9055.94	70.7747



- Refine until maximum relative eigenvalue error (est.) drops below tol$= 10^{-5}$
- Check error every $\mathrm{pause} + 1$ steps
- Can reduce cost by 70%

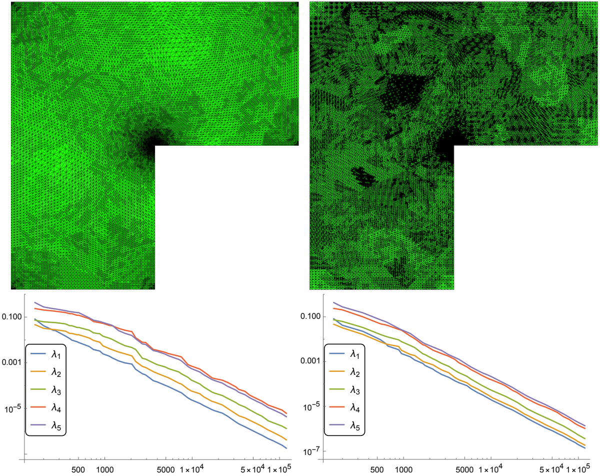
Illustrating Partial Fourier Expansion

$$\mathcal{L}v \doteq -\Delta v + cv \quad , \quad \mathcal{L}u = 1 \quad , \quad \mathcal{L}\psi = \lambda\psi$$



- Expansion using 13 “ground states” (middle), $\{\psi_j : j \in \{1, 2, 3, 4, 6, 7, 9, 10, 13, 16, 19, 27, 48\}\}$
- Expansion using first 50 eigenvectors (right)

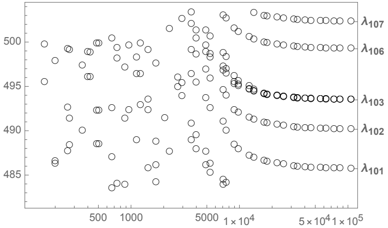
L Shape Domain, h-Adaptivity, $\mathcal{L} = -\Delta$ $\{\lambda_1, \cdots, \lambda_5\}$



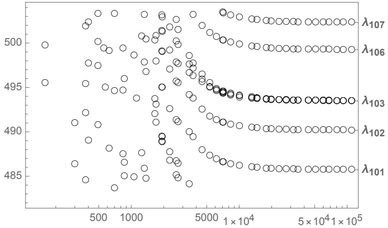
- Landscape refinement (left) and cluster refinement
- Optimal order of convergence in both cases
- Final errors for cluster refinement slightly smaller

L Shape Domain, h-Adaptivity, $\mathcal{L} = -\Delta$ $\lambda \in 50\pi^2 + [-10, 10]$

Landscape Refinement



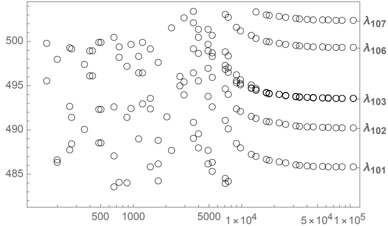
Cluster Refinement



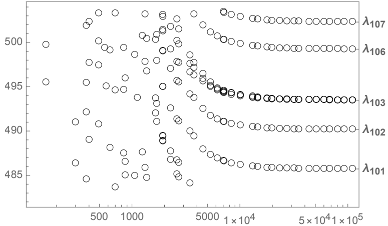
- Neither approach really resolves eigenpairs until $\sim$ 5000 DOF
- Cluster refinement driven by “poor information” before then
  - If it is doing a reasonable job early on, it is getting lucky
- Compare pure landscape refinement with cluster refinement and three mixed strategies that shift from landscape to cluster refinement
  - Shift at DOF $>$ 4000 (MR1), DOF $>$ 8000 (MR2), DOF $>$ 10000 (MR3)

L Shape Domain, h-Adaptivity, $\mathcal{L} = -\Delta$ $\lambda \in 50\pi^2 + [-10, 10]$

Landscape Refinement



Cluster Refinement



	DOF	e_{101}	e_{102}	e_{103}	e_{104}	e_{105}	e_{106}	e_{107}
CR	110123	4.30e-03	4.29e-03	4.02e-03	4.51e-03	5.11e-03	5.27e-03	5.18e-03
LR	106638	2.57e-02	2.40e-02	2.36e-02	2.42e-02	2.70e-02	2.61e-02	2.30e-02
MR1	104095	4.81e-03	4.70e-03	4.37e-03	4.99e-03	5.67e-03	5.76e-03	5.77e-03
MR2	115258	3.81e-03	3.77e-03	3.57e-03	4.03e-03	4.67e-03	4.67e-03	4.76e-03
MR3	104432	1.96e-03	2.07e-03	1.92e-03	2.06e-03	3.12e-03	2.42e-03	2.59e-03

- Reporting errors on finest mesh
- All five approaches yield optimal order convergence $\mathcal{O}(\mathrm{DOF}^{-2})$
- Switching from LR to CR latest ($\mathrm{DOF} > 10000$) is best in this case