The Unit Circle

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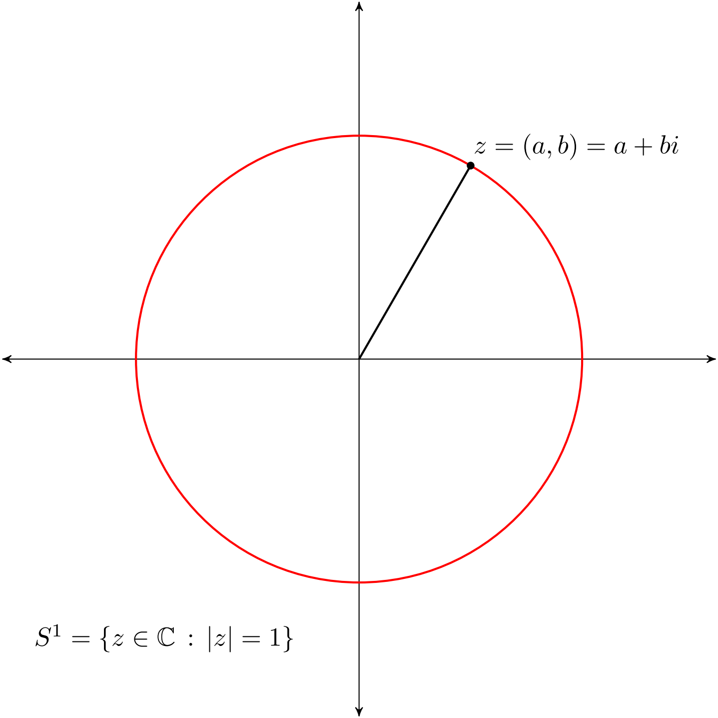
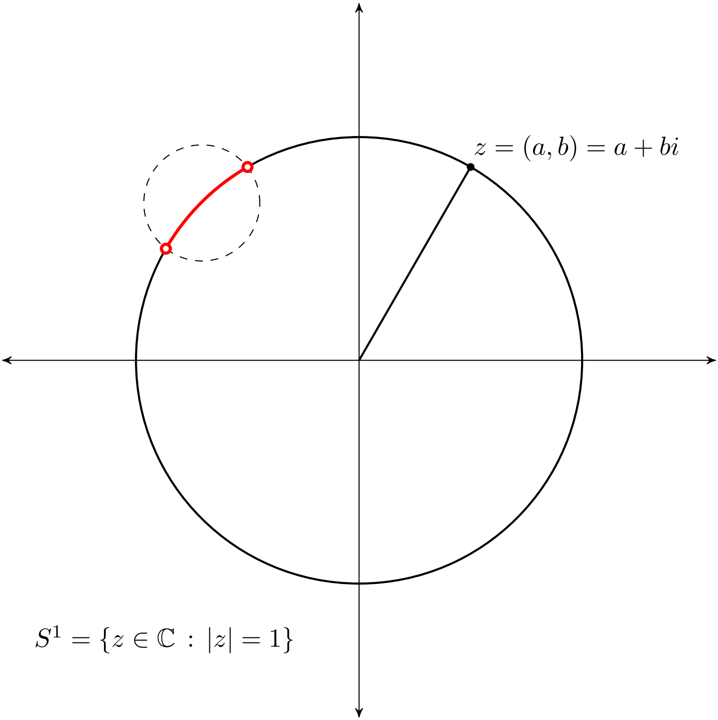
math *Club*


$z = (a, b) = a + bi$
$S^1 = \{z \in \mathbb{C} : |z| = 1\}$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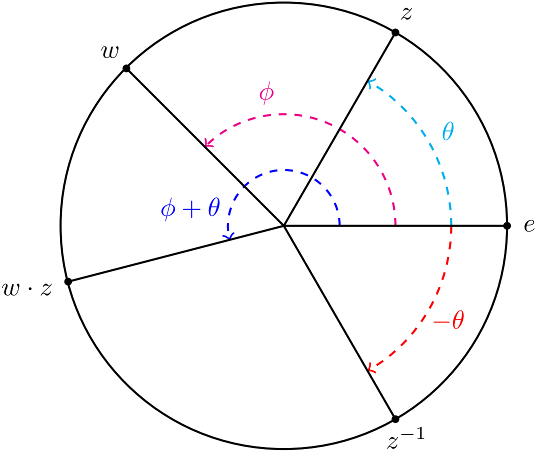
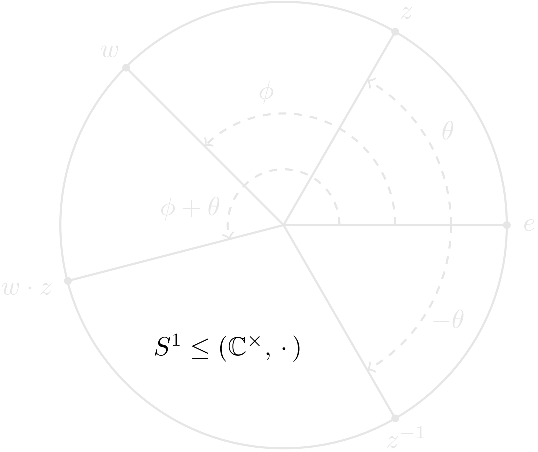


z
w
ϕ
θ
$\phi + \theta$
e
$w \cdot z$
$-\theta$
z^{-1}



$S^1 \leq (\mathbb{C}^\times, \cdot)$

The maps $\mu(w, z) = w \cdot z$ and $\iota(w) = w^{-1}$ are differentiable.

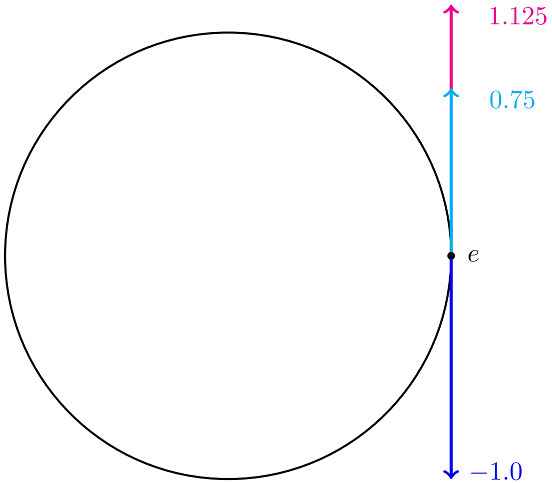
Proof. One says that "μ is differentiable" if the function $f(\phi, \theta) = \phi + \theta$ is differentiable. We check that each partial derivative is continuous at (ϕ, θ).

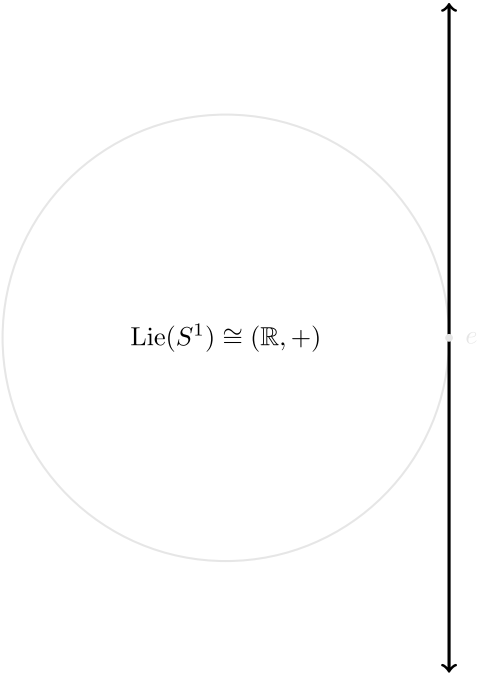
$$f_\phi = f_\theta = 1$$

For the map ι we take the ordinary derivative of $g(\phi) = -\phi$.

$$g'(\phi) = -1$$

Because S^1 is a topological space, a group, and both maps above are differentiable we call it a ***Lie group***.


1.125
0.75
e
−1.0


$\mathrm{Lie}(S^1) \cong (\mathbb{R}, +)$
e

The Lie algebra of a Lie group G is usually denoted by $\mathfrak{g}$. One studies $\mathfrak{g}$ and then returns to G with the ***exponential map***

$$\exp : \mathfrak{g} \to G$$

The map $\exp : \mathbb{R} \to S^1$ turns out to be

$$\exp(\theta) = e^{i\theta} = \cos\theta + i \sin\theta$$

A ***representation*** of a Lie group G in a vector space V is a homomorphism

$$\rho : G \to \mathrm{Aut}(V)$$

The map $\varphi : S^1 \to GL(2, \mathbb{R})$ is a representation of S^1 in $\mathbb{R}^2$ defined by

$$\varphi(z) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

There is a natural representation of G in $\mathfrak{g}$ written as

$$\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$$

where Ad stands for ***adjoint***.

For the unit circle each (unitary) irreducible representation is of the form

$$\rho_n(z) = z^n \quad n \in \mathbb{Z}$$

The set $\{\rho_n\} \cong \mathbb{Z}$ is called the ***dual group*** of S^1.

The Fourier transform of a function $f : \mathbb{R} \to \mathbb{C}$ is given by

$$\hat{f}(s) = \int_{-\infty}^{\infty} e^{2\pi i s t} f(t)\, dt\,, \quad s \in \mathbb{R}$$

For some $f : S^1 \to \mathbb{C}$ the formula is

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} e^{in\phi} f(\phi)\, d\phi\,, \quad n \in \mathbb{Z}$$

References

[1] Timothy Gowers, editor. *The Princeton Companion to Mathematics*. Princeton University Press, 2010.

[2] A. A. Kirillov. *Elements of the Theory of Representations*. Springer, 1976.

[3] Wikipedia. Circle group. `https://en.wikipedia.org/wiki/circle_group`, 2015.