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NOTE

A note on the formulation of the Maxwell equations for a macroscopic medium

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Abstract
In formulating the Maxwell equations for electrodynamics in the presence of a macroscopic continuous medium, it is common among standard textbooks to simply borrow expressions for the polarization charges and magnetization currents derived in electrostatics and magnetostatics, without much explicit justifications for using them in electrodynamics. Here we emphasize that these quantities must be introduced without resorting to results in statics, and must be derived directly from their definitions which remain valid in the general situation of electrodynamics.

Introduction

It is a very common approach, in standard popular electrodynamics texts (see, e.g. [1–3]), that the expressions for the polarization charge and magnetization current,

$$\rho_P = -\vec{\nabla} \cdot \vec{P}, \tag{1}$$

$$\vec{J}_M = \vec{\nabla} \times \vec{M}, \tag{2}$$

are first derived under the special condition of electrostatics and magnetostatics, and then applied, without explicit justifications, to the most general *electrodynamic* case leading finally to the source pair of Maxwell equations in a medium as follows:

$$\vec{\nabla} \cdot (\varepsilon_0 \vec{E} + \vec{P}) \equiv \vec{\nabla} \cdot \vec{D} = \rho \tag{3}$$

and

$$\vec{\nabla} \times \left(\frac{1}{\mu_0} \vec{B} - \vec{M} \right) - \frac{\partial \vec{D}}{\partial t} \equiv \vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J} \tag{4}$$

respectively. Note that with the application of the definitions for both $\vec{P}$ and $\vec{M}$ in terms of the dipole moment densities, the results in (1) and (2) can be derived from the following expressions for the scalar and vector potentials, as usually done in standard texts [1–3],

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\vec{P}(\vec{r}\,')\cdot(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\,\mathrm{d}^3x' = \frac{1}{4\pi\varepsilon_0}\int \frac{-\vec{\nabla}'\cdot\vec{P}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,\mathrm{d}^3x' \tag{5}$$

and

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{M}(\vec{r}\,')\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\,\mathrm{d}^3x' = \frac{\mu_0}{4\pi}\int \frac{\vec{\nabla}'\times\vec{M}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,\mathrm{d}^3x'. \tag{6}$$

On the other hand, the results in (5) and (6) are restricted to be valid *only* in the case of statics and fail in the general case of electrodynamics due mainly to the neglect of retardation effects. This then leads to the following puzzling question for students in an introductory class of electromagnetism: how can the results in (1) and (2) be applied to formulate the electrodynamic equations in (3) and (4) if they are derived *only* from (5) and (6), whose validity is limited to cases of electro- and magnetostatics?

It is the purpose of this note to justify the results in (1) and (2) by deriving them *directly* from the definitions of the polarization and magnetization vectors *without* using the results in (5) and (6). Our approach here is similar to that of Landau and Lifshitz [4], except that the derivations in [4] were done within the context of electrostatics and magnetostatics which is quite unnecessary. In addition, our derivations here are quite similar for both the results in (1) and (2) via the establishment of a mathematical identity, with the neutrality condition in deriving (1) only implicitly imposed in our definition of the volume of integration (see below) which can as well fall inside the dielectric medium. Thus our approach is slightly different from but equivalent to that of [4] where the authors have based their derivation explicitly on the neutrality condition.

The dielectric medium

We start with the following expressions for the electric and magnetic dipole moments:

$$\vec{p} = \int_V \vec{P}\,\mathrm{d}^3x = \int_V \vec{r}\rho_P\,\mathrm{d}^3x + \oint_S \vec{r}\sigma_P\,\mathrm{d}a, \tag{7}$$

and

$$\vec{m} = \int_V \vec{M}\,\mathrm{d}^3x = \frac{1}{2}\int_V \vec{r}\times\vec{J}_M\,\mathrm{d}^3x + \frac{1}{2}\oint_S \vec{r}\times\vec{K}_M\,\mathrm{d}a. \tag{8}$$

Note that the definitions in (7) and (8) are good in general including the electrodynamic case [1], and we have included explicitly contributions from both the volume- and surface-induced charges and currents. Our following derivations are therefore not limited to only the static case, as occurs using (5) and (6) [1–3]. Note also that the volume of integration in (7) and (8) can fall within the dielectric, but must contain an integral number of the induced (molecular) dipole moments to make the definitions of the polarization and magnetization vectors in terms of the dipole density meaningful. In other words, the boundary (S) of the volume (V) is not allowed to separate the charges in a molecular dipole. The situation is similar to the case with the definition of charge density where the charge contained in a small volume must be an integral amount of the elementary charge (e) since such a charge cannot be split. Thus, the volumes of integration in (7) and (8) automatically imply the neutrality condition assuming that no external free charges are present.

To demonstrate how (1) follows from (7), we consider the *identity*

$$\int_V \vec{\nabla}\cdot(x_i\vec{P})\,\mathrm{d}^3x = \int_V (x_i\vec{\nabla}\cdot\vec{P} + \vec{P}\cdot\vec{\nabla}x_i)\,\mathrm{d}^3x = \int_V (x_i\vec{\nabla}\cdot\vec{P} + P_i)\,\mathrm{d}^3x, \tag{9}$$

which, with the application of the divergence theorem, implies [4]

$$\int_V \vec{P}\,\mathrm{d}^3x = \int_V \vec{r}(-\vec{\nabla}\cdot\vec{P})\,\mathrm{d}^3x + \oint_S \vec{r}(\vec{P}\cdot\hat{n})\,\mathrm{d}a. \tag{10}$$

Equations (7) and (10) then imply the following result:

$$\int_V \vec{r}(\rho_P + \vec{\nabla}\cdot\vec{P})\,\mathrm{d}^3x + \oint_S \vec{r}(\sigma_P - \vec{P}\cdot\hat{n})\,\mathrm{d}a = 0. \tag{11}$$

Since (11) is valid for *any* arbitrary V (except for the restriction described above) and the corresponding boundary S (hence also arbitrary)[1], together with the arbitrariness in the choice of the origin of $\vec{r}$, we conclude that (1) must be valid with the corresponding surface charge density given by $\sigma_P = \vec{P}\cdot\hat{n}$. Note that the surface charge term necessary emerges even if V is enclosed inside the medium as is clear from equation (7).

The physical meaning of the general validity of (1) is clear following the arguments in [4]. By imposing explicitly the neutrality condition in the absence of external free charges, we have $\int_V \rho_P\,\mathrm{d}^3x + \oint_S \sigma_P\,\mathrm{d}a = 0$, which then guarantees the volume integral to be convertible to a surface term. Hence through the divergence theorem, it is then always possible to express the polarized charge density by the divergence of a vector, determined as $-\vec{P}$ from our result in (11). Thus our approach can be regarded as an illustration of the self-consistency of the formulation in [4] which is valid in the general electrodynamic situation, and hence more justified than the one found in most popular texts [1–3].

The magnetic medium

Next we justify (2) in a way similar to the above for polarization charge, by deriving again the following *mathematical identity* without using any result from magnetostatics:

$$\int_V \vec{M}\,\mathrm{d}^3x = \frac{1}{2}\int_V \vec{r}\times(\vec{\nabla}\times\vec{M})\,\mathrm{d}^3x + \frac{1}{2}\oint_S \vec{r}\times(\vec{M}\times\hat{n})\,\mathrm{d}a. \tag{12}$$

To establish the result in (12), we consider a component of the volume integral on the right-hand side:

$$\begin{aligned}
\int_V [\vec{r}\times(\vec{\nabla}\times\vec{M})]_i\,\mathrm{d}^3x &= \int_V \varepsilon_{ijk}x_j\varepsilon_{k\ell m}\partial_\ell M_m\,\mathrm{d}^3x \\
&= \int_V (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell})x_j\partial_\ell M_m\,\mathrm{d}^3x \\
&= \int_V (x_j\partial_i M_j - x_j\partial_j M_i)\,\mathrm{d}^3x \\
&= \int_V (-M_j\partial_i x_j + M_i\partial_j x_j)\,\mathrm{d}^3x + \int_V [\partial_i(x_jM_j) - \partial_j(x_jM_i)]\,\mathrm{d}^3x \\
&= \int_V (-M_j\delta_{ij} + 3M_i)\,\mathrm{d}^3x + \oint_S (\vec{r}\cdot\vec{M})n_i\,\mathrm{d}a - \oint_S (\vec{r}\cdot\hat{n})M_i\,\mathrm{d}a \\
&= 2\int_V M_i\,\mathrm{d}^3x - \oint_S [\vec{r}\times(\vec{M}\times\hat{n})]_i\,\mathrm{d}a,
\end{aligned} \tag{13}$$

[1] We thank an anonymous referee for insisting this condition which led to the improvement of a previous version of our note.

where we have applied twice the divergence theorem in various forms [1]. Hence (12) is an identity without having to resort to any results from magnetostatics. It is then obvious from the comparison of (8) and (12), again for *arbitrary* V and S, that the magnetization current is indeed given by the result in (2) with the corresponding surface current given by $\vec{K}_M = \vec{M} \times \hat{n}$, and the results are clearly valid in the general electrodynamic case due to the general validity of (8) and (12). Note that our derivation of (2) is similar to that given in [4] except that [4] treated this only within the context of magnetostatics which is quite unnecessary. With the justification of the results in (1) and (2) without resorting to electrostatics and magnetostatics, the general formulation of Maxwell's equations as in (3) and (4) for the electrodynamics in a dielectric medium can then be justified without ambiguity.

It is of interest to ask why the magnetostatic result in (6) can turn out to lead to the correct result in (2) which is generally valid in electrodynamics. The reason is that the current component which enters into (8) must be divergenceless even though currents are not divergenceless in general in the dynamic case. This will enable the magnetostatic result obtained from (6) which assumes divergenceless currents to lead to the same result as obtained in electrodynamics. To show the details explicitly, we illustrate with the volume current and start by recalling the Helmhotz theorem which allows the current to be decomposed into a divergenceless transverse component and a curlless longitudinal component as follows (let $\vec{J}_M \to \vec{J}$):

$$\vec{J} = \vec{J}_t + \vec{J}_\ell, \qquad \text{where} \quad \vec{\nabla} \cdot \vec{J}_t = 0 \qquad \text{and} \qquad \vec{\nabla} \times \vec{J}_\ell = 0. \tag{14}$$

Next we show that the longitudinal component will not survive when plugged into the definition of the magnetic moment in (8). This is clear since by writing $\vec{J}_\ell = \vec{\nabla}\lambda$, λ being an arbitrary scalar function, we have the following integral

$$\begin{aligned} \int \vec{r} \times \vec{J}_\ell \, \mathrm{d}^3x &= \int \vec{r} \times \vec{\nabla}\lambda \, \mathrm{d}^3x \\ &= \int [\lambda \vec{\nabla} \times \vec{r} - \vec{\nabla} \times (\lambda \vec{r})] \, \mathrm{d}^3x \\ &= \oint_S \lambda \vec{r} \times \hat{n} \, \mathrm{d}a = 0. \end{aligned} \tag{15}$$

Hence from (8), the result in (15) implies that only the divergenceless transverse component of the current will contribute to the magnetization even in the non-steady-state situation when $\vec{\nabla} \cdot \vec{J} \neq 0$. It is of interest to note that the direction of $\vec{J}_\ell$ is along the normal at the surface, where as the surface magnetization current $\vec{K}_M$ is always tangential to it and hence capable of forming closed current loops. Thus the physical reason for the contribution to magnetization to come exclusively from the transverse current is due to the formation of closed loops by this current. In a similar vein, one can also 'understand' why the correct result in (1) can also be derived from applying only the electrostatic result in (5). This is due to the overall neutrality of the bound charges which is valid in electrodynamics as well as in electrostatics [4]. However, deriving the results only in statics and then applying them directly to electrodynamics will likely leave students to puzzle about their validities.

Conclusion

In summary, we have shown in the above how the results in (1) and (2) can be established in a general way so that they can be used to formulate the general Maxwell equations for a ponderable medium. Occasionally, one also finds this correct 'direct derivation' from the definitions adopted by some authors in a non-mathematical (e.g. graphical) approach [3, 5, 6].

But the one presented here is mathematically rigorous and valid for electrodynamics in general. The fact that the results in (1) and (2) have been established only in electrostatics and magnetostatics in some standard texts [1–3] may have been overlooked for some time. Our clarification above should help to put them on a rigorous base and should therefore be useful for instructors in courses of classical electrodynamics.

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