A surprising twist to a simple capacitor problem

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Abstract. We show that the standard solution to the standard problem of finding the force with which a dielectric is pulled into a parallel plate capacitor still offers a surprise.

Zusammenfassung. Es wird gezeigt, daß die typische Lösung für das Auffinden der Kraft, mit der ein Dielektrikum in einen Parallelplattenkondensator gezogen wird, immer noch Überraschungingen bereith ält.

In physics we are taught (and therefore we teach) that certain ways or methods of looking at problems are extremely powerful. As a consequence one is able to solve otherwise seemingly intractable problems. One such method is the use of energy (or virtual work) in, for instance, mechanics problems and the other is the use of symmetry. In the following problem these two techniques are in apparent conflict.

Consider a rectangular parallel plate capacitor of width w, length l, and a separation between the plates of d. It is assumed that $d \ll l$ and $d \ll w$. Suppose that the space between the plates is just filled with a solid dielectric (size $d \times w \times l$) with a relative dielectric constant of ε_r. The capacitance is given by

$$C = \varepsilon_r \varepsilon_0 \frac{wl}{d}. \tag{1}$$

The capacitor is charged to a voltage V and then is disconnected from the battery (if the battery is not disconnected there is small complication but the results are no different, see Kip 1969). Next we pull the dielectric a distance x lengthwise from its original position. The force we have to exert can easily be calculated because we can find the potential energy stored in the capacitor as a function of x (see, e.g. Feynman *et al* 1964). The potential energy, $U(x)$, is given by

$$U = Q^2 \left[2\left(\varepsilon_0 \frac{xw}{d} + \varepsilon_r \varepsilon_0 \frac{(l-x)w}{d} \right) \right]^{-1} \tag{2}$$

and thus the force, $F(x)$, exerted on the dielectric by the electric field is given by

$$F = -\frac{\mathrm{d}U}{\mathrm{d}x} = -Q^2 \varepsilon_0 (\varepsilon_r - 1) \frac{w}{d}$$

$$\times \left[2\left(\varepsilon_0 \frac{xw}{d} + \varepsilon_r \varepsilon_0 \frac{(l-x)w}{d} \right)^2 \right]^{-1}. \tag{3}$$

As a check of this result we can calculate the force when x equals zero since we know that, because of symmetry, the force should be equal to zero. Substituting $x = 0$ in equation (3) leads to a non-zero value for the force. Thus we apparently have a conflict. Mathematically, one can show that taking the derivative in equation (3) is not correct. We have plotted the potential energy in figure 1 (curve A) for both positive and negative x. The latter corresponds to pushing the dielectric slab rather than pulling it. It is clear that $U(x)$ in equation (2) is not differentiable at $x = 0$ and hence $F(0)$ cannot be given by equation (3).

The explanation lies in the fringing fields at the edges of the capacitor plates which we have ignored so far. In fact, if there were no fringing field, the force on the dielectric would be equal to zero, irrespective of x (see Griffiths 1989). Clearly, calculating the force on the dielectric from the fringing field is not trivial and by using the potential energy one circumvents this rather cleverly. However, at $x = 0$ the virtual work method gives an erroneous result which is seldom pointed out. Calculating $U(x)$ correctly is not the goal (as alluded to earlier, neither is this easy and neither is this illuminating), what we will give though is a 'physical proof'.

The argument that $F(0)$ should be zero if the fringing field is taken into consideration is as follows. It is not difficult to see that for a small displacement Δx away from $x = 0$, the potential energy of the system should not be changed since the part of the slab in the fringing field just compensates for that part which is 'absent from' the inside of the capacitor. This is

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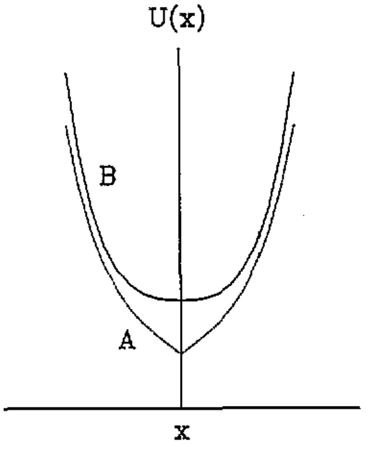

Figure 1. Schematic diagram for the potential energy.

guaranteed since the proximity fringing field must be equal to the field just inside the capacitor from Faraday's law (Halliday and Resnick 1988). Hence, we conclude that $U(0 \pm \Delta x) = U(0)$ or

$$F(0) = \lim_{\Delta x \to 0} \frac{U(0 + \Delta X) - U(0)}{\Delta x} = 0$$

only if fringing effects are included. In addition, the potential energy versus x graph including the fringing effects should look something like curve B in figure 1.

The capacitor problem has been presented on several occasions to people of all levels of sophistication and initially, they were invariably surprised that the 'standard' solution breaks down in the limiting case of x approaching zero. We feel that this problem is fascinating from the point of view of general problem solving techniques and also because it shows that the fringing fields are sometimes crucially important. As an aside it should be mentioned that the fact that the solution should break down is also evident from the fact that when x approaches zero, the conditions inherent to equation (1) fail to be satisfied.

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