MTH 322
SAMPLE FINAL

1. Consider the initial value problem

$$u_t + (1+u)u_x = 0, \quad -\infty < x < \infty, \quad t > 0$$

$$u(x,0) = \begin{cases} 0.5 & \text{x} \leq -1, \\ 0 & -1 < x \leq 0. \\ 1 & x > 0. \end{cases}$$

   (a) Use the method of characteristics to show that the solution is initially a combination of a rarefaction wave and a shock wave.
   (b) Find the speed of propagation of the shock wave part of this solution.
   (c) Determine the time at which the rarefaction wave will start interacting with the shock wave and set up (but do not solve it) the corresponding initial vale problem for the new shock wave.

2. Determine, using the viscosity method, if the initial condition

$$u(x,0) = \begin{cases} 1 & \text{x} \leq 0, \\ 2 & x > 0. \end{cases}$$

will propagate as a shock wave solution of the differential equation

$$u_t - uu_x = 0, \quad -\infty < x < \infty, \quad t > 0.$$

3. Find the dispersion relation for the wave train solution of

$$u_{tt} - u_{xx} + u_{xxxx} = 0.$$

Determine if the equation is dispersive. Remember, both the wave number and the circular frequency are assumed to be positive.

4. Find a rarefaction wave solution, i.e., $u(x,t) = g(\frac{x-a}{t-b})$, of the following initial value problem:

$$u_t + (u^2 + 1)u_x = 0, \quad -\infty < x < \infty, \quad t > 0$$

$$u(x,0) = \begin{cases} 0 & \text{x} \leq -1, \\ \\ 1 & x > -1. \end{cases}$$

5. Use characteristics to construct an xt-diagram representation for the d'Alembert solution of the semi-infinite problem

$$u_{tt} = u_{xx}, \quad 0 < x < \infty, \quad 0 < t < 2,$$

$$u(x,0) = \begin{cases} 0 & 0 < x < 1, \\ \\ 1 & 1 < x < 2, \\ \\ 0 & otherwise, \end{cases}$$

$$u_t(x,0) = 0, \quad u(0,t) = 0.$$

6. Find the traveling wave solution of the differential equation

$$u_t + u_x + u = 0, \quad -\infty < x < \infty, \quad t > 0,$$

and determine the speed(s) of propagation of these waves.

7. Consider the rarefaction wave solution from Problem 1(a). Show that it satisfies the entropy condition

$$\frac{u(x+h,t)-u(x,t)}{h} \leq \frac{E}{t}$$

for some positive constant E? Explain your answer.