Homework 9
Solutions

- 21.1 Calculating the x and t partial derivatives of the substitution

$$U = v_1(1 - \frac{2u}{u_1})$$

we obtain

$$U_t = -2v_1\frac{u_t}{u_1}, \quad U_x = -2v_1\frac{u_x}{u_1}, \quad U_{xx} = -2v_1\frac{u_{xx}}{u_1}.$$

Therefore,

$$u_t = -\frac{u_1}{2v_1}U_t, \quad u_x = -\frac{u_1}{2v_1}U_x, \quad u_{xx} = -\frac{u_1}{2v_1}U_{xx}.$$

Substituting these derivatives and the function U into our equation and dividing by the common factor of $-\frac{u_1}{2v_1}$, we obtain

$$U_t + UU_x = rU_{xx}.$$

- 22.3 You should be able to show that the rarefaction wave modeling the traffic after the light turn green is

$$u(x,t) = \begin{cases} u_1 & x \le -v_1 t, \\ \frac{u_1}{2}\left(1 - \frac{x}{v_1 t}\right) & -v_1 t < x \le v_1 t, \\ 0 & x > v_1 t. \end{cases}$$

- 24.3 As we discussed in class, the rarefaction wave of this IVP does not satisfy the entropy condition at the vicinity of $x = 0$. However, see below the solution to the Problem 24.2, showing a rarefaction wave solution which does satisfy the entropy condition.

- 24.2 The corresponding rarefaction wave solution is

$$u(x,t) = \sqrt{\frac{x}{t}}.$$

Note that it is valid only for $t < x < 4t$ as its most left characteristic is $x = t$ while the most right one is $x = 4t$.

Now, looking at the secant of the rarefaction wave we have that:

$$\frac{\sqrt{\frac{x+h}{t}} - \sqrt{\frac{x}{t}}}{h} = \frac{(\sqrt{\frac{x+h}{t}} - \sqrt{\frac{x}{t}})(\sqrt{\frac{x+h}{t}} + \sqrt{\frac{x}{t}})}{h(\sqrt{\frac{x+h}{t}} + \sqrt{\frac{x}{t}})} = \frac{1}{\sqrt{t}(\sqrt{x+h} + \sqrt{x})}.$$

Taking into account the fact that $h > 0$ and that $t < x$, one is able to show that

$$\frac{1}{\sqrt{t}(\sqrt{x+h} + \sqrt{x})} \leq \frac{1}{2\sqrt{t}\sqrt{x}} < \frac{1}{2(\sqrt{t})^2} = \frac{1}{2t}$$

proving that the given rarefaction wave satisfies the entropy condition with $E = \frac{1}{2}$.