Homework 8
Solutions

- 22.2 We are looking for the solution

$$u(x,t) = g(\frac{x}{t}).$$

After substituting it into our conservation law $u_t + u^2 u_x = 0$ we get

$$-\frac{x}{t^2} g' + g^2 g' \frac{1}{t} = 0.$$

As the constant solution is not acceptable as a rarefaction wave we finally arrive at the relation

$$g(\frac{x}{t}) = \sqrt{\frac{x}{t}}.$$

Note that, based on the given initial condition, the most left characteristic of the rarefaction wave is $x = t$ while the most right characteristic is $x = 4t$. This means that the solution takes the following form:

$$u(x,t) = \begin{cases} 1, & x \leq t, \\ \sqrt{\frac{x}{t}}, & t < x \leq 4t, \\ 2, & x > 4t. \end{cases}$$

- 22.5 The line $x = v_1 t$ represent the front of moving cars while the line $x = -v_1 t$ indicates the position (in time) where the cars are still stopped. In other words, there are no cars in front of the line $x = v_1 t$, the density of cars increases gradually between $x = v_1 t$ and $x = -v_1 t$, and the density of cars is maximum (they are not moving) at and beyond the line $x = -v_1 t$.

- 23.4 To identify the corresponding characteristics, note that

$$c(u) = v_1 \left(1 - 2\frac{u}{u_1}\right).$$

Therefore,

$$c(0) = v_1, \quad c(u_1) = -v_1, \quad c(\frac{u_1}{2}) = 0.$$

This implies that the corresponding characteristics are:

$$x = v_1 t + x_0, \quad x = -v_1 t + x_0, \quad x = x_0.$$

To find the rarefaction wave solution, see the previous problem. However, to find the shock wave (back-shock) note first that the flux

$$\phi(u) = v_1 u - \frac{v_1 u^2}{u_1}.$$

The Rankine-Hugoniot condition is therefore

$$\frac{dx}{dt} = -\frac{1}{4} v_1 u_1, \quad x(0) = -L$$

and the shock curve is

$$x = -\frac{1}{4} v_1 u_1 t - L.$$

- Problem 11 from the Review set: To find the solution to the given initial-value problem using the artificial viscosity method, we replace the given equation (conservation law) by the equation

$$u_t + 2uu_x = \epsilon u_{xx}$$

where $\epsilon > 0$. Our objective is to find a traveling wave $u(x,t) = f(x - ct)$ subject to the following conditions:

$$\lim_{z \to \infty} f(z) = 1, \ \lim_{z \to -\infty} f(z) = 2, \ \lim_{z \to \pm\infty} f'(z) = 0$$

that is, mimicking our original initial condition at infinities. Here, $z = x - ct$. Substituting the formula for the traveling wave into the differential equation, we obtain the following ordinary differential equation for the shape f and the speed of propagation c:

$$-cf' + 2ff' = \epsilon f''.$$

Integrating this equation once we obtain

$$-cf + f^2 = \epsilon f' + k$$

where k is the constant of integration. Considering the above listed conditions at infinities we obtain the following equations for the speed c and the constant k:

$$\begin{cases} -c+1=k \\ -2c+4=k. \end{cases}$$

Solving this system of linear equations we obtain $c=3$ and $k=-2$. We are now left with solving the following ordinary differential equation for the shape f:

$$f^2-3f+2=\epsilon f'.$$

This is a separable first-order ordinary differential equation. Using the method of partial fractions we obtain

$$\int \frac{1}{f-2}df - \int \frac{1}{f-1}df = \frac{1}{\epsilon}\int dz.$$

This gives us

$$\ln \frac{f-2}{f-1} = \frac{1}{\epsilon}z + A,$$

where A is the integration constant. Equivalently, this leads to the relation

$$\frac{f-2}{f-1} = Ae^{\frac{x-3t}{\epsilon}}.$$

It should now be easy to see that

$$\text{if } x > 3t \ \lim_{\epsilon\to 0} f(z) = 1$$

$$\text{if } x < 3t \ \lim_{\epsilon\to 0} f(z) = 2$$

proving that our unique solution is the forward shock

$$u(x,t) = \begin{cases} 2 & x < 3t, \\ 1 & x > 3t. \end{cases}$$