Homework 7
Solutions

- 18.4 The equation of a characteristic $x(t)$ for the differential equation $u_t + u^2 u_x = 0$ is
$$\frac{dx}{dt} = u^2(x(t), t).$$
As the solution is constant on characteristics, the above equation reduces to
$$\frac{dx}{dt} = u_0^2(x_0) = \frac{1}{(1+x_0^2)^2}$$
where x_0 denotes the initial point. Therefore, the characteristic is given by
$$x = \frac{1}{(1+x_0^2)^2} t + x_0.$$
Given the characteristic starting at $(x_0, 0)$, the critical time at which it will meet with another characteristic is
$$t_b = \frac{-1}{2\frac{1}{1+x_0^2}\frac{d}{dx_0}\left[\frac{1}{(1+x_0^2)^2}\right]} = \frac{(x_0^2+1)^3}{4x_0}.$$
To find the critical time at which the shocks will develop one must find the minimum of t_b as a function of x_0. A simple differentiation shows that the function t_b has two critical points
$$x_0 = \pm\frac{1}{\sqrt{5}}.$$
Substituting these two values to the formula for the breaking time t_b yields that
$$t_b \simeq 0.966.$$

- 19.2 The flux function of the conservation law $u_t + u^2 u_x = 0$ is
$$\phi(u) = \frac{1}{3}u^3.$$

Thus, the Rankine-Hugoniot condition takes the form

$$\frac{dx_s}{dt} = \frac{1}{3}\left[(u^+)^2 + (u^-)^2 + u^+u^-\right], \quad x_s(0) = 0,$$

where $u^+ = 1$ and $u^- = 2$. This yields the shock curve

$$x_s = \frac{7}{3}t.$$

Consequently, the solution

$$u(x,t) = \begin{cases} 2 & x \leq \frac{7}{3}t \\ 1 & x > \frac{7}{3}t. \end{cases}$$

- 20.2 If the incoming traffic's speed is 15 miles per hour then, according to the formula

$$v = v_1(1 - \frac{u}{u_1}),$$

the density of the incoming traffic is $u_0 = 200$ cars per mile. This implies, as the flux

$$\phi(u) = \frac{v_1}{u_1}(uu_1 - u^2),$$

that the back-shock propagates with the speed of -30. Therefore the solution is:

$$u(x,t) = \begin{cases} 200, & x \leq -30t \\ 300, & x > -30t. \end{cases}$$

- 20.3

  - (a) I see no fundamental difference.
  - (b) The flux

$$\phi(u) = uv = uv_1\left(1 - \frac{u^2}{u_1^2}\right).$$

  Therefore, the Rankine-Hugoniot condition takes the form

$$\frac{dx_s}{dt} = \frac{[\phi(u)]}{[u]} = v_1 - \frac{v_1}{u_1^2}\left((u^-)^2 + (u^+)^2 + u^-u^+\right), \quad x_s(0) = 0.$$

As $u^+ = u_1$ and $u^- = u_0$ the shock curve is the straight line

$$x_s = -\frac{v_1 u_0}{u_1}\left(1 + \frac{u_0}{u_1}\right) t.$$

Consequently,

$$u(x,t) = \begin{cases} u_0, & x \leq -\frac{v_1 u_0}{u_1}\left(1 + \frac{u_0}{u_1}\right) t, \\ u_1, & x > -\frac{v_1 u_0}{u_1}\left(1 + \frac{u_0}{u_1}\right) t. \end{cases}$$