Homework 6
Solutions

- 15.2
  - (a)
  $$u_t + cu_x = u_t + (cu)_x = 0,$$
  - (b)
  $$u_t + uu_x - Du_{xx} = u_t + \left(\frac{1}{2}u^2 - Du_x\right)_x = 0,$$
  - (c)
  $$u_t + uu_x + u_{xxx} = u_t + \left(\frac{1}{2}u^2 + u_{xx}\right)_x = 0.$$

- 16.5 As the corresponding flux function takes the form
  $$\phi(u) = uv = ku\ln(\frac{u_1}{u}),$$
  the differential equation $u_t + \phi(u)_x = 0$ becomes
  $$u_t + k[\ln(\frac{u_1}{u}) - 1]u_x = 0.$$
  As far as drivers' behavior is concerned, notice that this model, in contrast to the other traffic flow model, has no speed limit. Thus, the velocity $k\ln(\frac{u_1}{u})$ approaches ∞ when the density of cars u approaches 0. In contract, according to the other model, the drivers will never exeat the maximum velocity v_1.

- 17.3 The characteristic are strait lines $x = 2t + a$, and the solution is constant on characteristics. To determine the solution we consider two possibilities either a characteristic intersect the initial condition, the x-axis, or it intersects the boundary condition, the t-axis.
  - (a) If a characteristic starts at $(x_0, 0)$ then its equation takes the form $x = 2t + x_0$ and $x_0 = x - 2t$. Therefore, if $x > 2t$, that is for the points (x, t) such that the corresponding characteristics intersects the x-axis,
  $$u(x,t) = u_0(x_0) = u_0(x - 2t) = 0$$
  as $u(x, 0) = 0$.

– (b) On the other hand, when a characteristic starts at $(0, t_0)$ its equation is $x = 2t - 2t_0$ and $t_0 = t - \frac{x}{2}$. Hence, as the solution is still constant on characteristics

$$u(x,t) = u(0,t_0) = u(0, t - \frac{x}{2}) = \frac{t - \frac{x}{2}}{1 + \left(t - \frac{x}{2}\right)^2},$$

as

$$u(0,t) = \frac{t}{1+t^2}.$$

- 17.8 As the equation of the characteristic starting at x_0 has the form

$$x = e^{-\frac{1}{x_0}} t + x_0,$$

the characteristic passing through $(x,t) = (1,2)$ takes the form

$$1 = 2e^{-\frac{1}{x_0}} + x_0.$$

It is impossible to solve this nonlinear equation exactly. One possible way to get an approximate solution is to approximate the exponential function by a power series. For example, if one approximates the power series by a linear function of its argument $-1/x_0$ one will be able to solve the second-order equation for x_0.