Homework 4

Solutions

- 10.3 See solutions 2 document.
- 10.6 See solutions 2 document.
- 11.3
  - (a) We are looking for a solution or a combination of solutions

$$u_k(x,t) = [A_k \cos k\pi t + B_k \sin k\pi t] \sin k\pi x.$$

Given the initial conditions $u(x,0) = 10 \sin \pi x$ and $u_t(x,0) = 0$, we must select the coefficients A_k and B_k so as to much the given initial conditions. To this end, note that $u_k(x,0) = A_k \sin k\pi x$ and it is equal to the given initial condition if $k = 1$ and $A_1 = 10$. Calculating now the time derivative of the expected solution, we obtain that

$$\frac{\partial}{\partial t} u_1(x,t) = [10 \cos \pi t + B_1 \sin \pi t] \sin \pi x.$$

It vanishes at $t = 0$ and all x only if $B_1 = 0$. This implies that the solution to the given initial-value-boundary problem is

$$u(x,t) = 10 \cos \pi t \sin \pi x.$$

  - (b) I leave it to you.
  - (c) As in the previous case, note that

$$u_k(x,0) = A_k \sin k\pi x = \sin 4\pi x.$$

Therefore, $k = 4$ and $A_4 = 1$. Consequently,

$$\frac{\partial}{\partial t} u_4(x,0) = 4\pi B_4 \sin 4\pi x = 2 \sin 4\pi x.$$

Thus, $B_4 = \frac{1}{2\pi}$. Finally, the solution takes the form

$$u_4(x,t) = [\cos 4\pi t + \frac{1}{2\pi} \sin 4\pi t] \sin 4\pi x.$$

- 11.5 Straightforward.

- 13.1 Indeed, assume that $u(x,t)$ and $v(x,t)$ are two solutions of the wave equation satisfying the given boundary conditions. In particular, $u(0,t) = v(0,t) = 1$. Taking into account the fact that the wave equation is linear consider $w(x,t) = u(x,t) + v(x,t)$ as a potential solution. However, $w(x,t)$ does not satisfy the boundary conditions as $w(0,t) = u(0,t) + v(0,t) = 2 \neq 1$. This shows that the principle of superposition does not apply when the boundary conditions are not homogeneous.

- 13.3(a) Let us consider the composite standing wave

$$u(x,t) = \sum_{k=1}^{N} (A_k \cos k\pi t + B_k \sin k\pi t) \sin k\pi x.$$

As it should be easy to see both boundary conditions are satisfied. In order to find the solution satisfying the given set of initial conditions let us evaluate

$$u(x,0) = \sum_{k=1}^{N} (A_k \cos k\pi 0 + B_k \sin k\pi 0) \sin k\pi x = \sum_{k=1}^{N} A_k \sin k\pi x.$$

In order to satisfy the condition that

$$u(x,0) = 10 \sin \pi x + 3 \sin 4\pi x$$

we must require that

$$A_1 = 10, A_4 = 3$$

while any other $A_k = 0$. Differentiating the solution $u(x,t)$ with respect to t and evaluating it at $t = 0$, we obtain

$$u_t(x,0) = \sum_{k=1}^{N} k\pi B_k \sin k\pi x.$$

It vanishes only if all $B_k = 0$. Therefore

$$u(x,t) = 10 \cos \pi t \sin \pi x + 3 \cos 4\pi t \sin 4\pi x$$

is the solution satisfying the given set of initial conditions.