Homework 3
Solutions

- 7.4
  - (a) Elementary, if you follow the sequence of indicated steps.
  - (b) Differentiate the first equation by t and multiply by L to get

    $$Li_{xt} + CLv_{tt} + GLv_t = 0.$$

    Next, differentiate the second equation by x and subtract from the above equation. This gives you

    $$LCv_{tt} + LGv_t - v_{xx} - Ri_x = 0.$$

    Eliminate the term Ri_x using the first original equation. You finally obtain that

    $$LCv_{tt} + (LG + RC)v_t - v_{xx} + RGv = 0.$$

- 8.2 Substitute the given function $u(x,t) = \cos t \sin x$ into the differential equation to show that it is indeed satisfied. Once you know that the given function $u(x,t)$ is a solution you know both corresponding initial values:

  $$u(x,0) = \sin x, \quad u_t(x,0) = 0.$$

  Hence, using the d'Alembert form of the solution solution we have that

  $$u(x,t) = \frac{1}{2}[\sin(x-t) + \sin(x+t)]$$

  indicating the left and the right moving waves.

- 8.5 Given the initial conditions

  $$u(x,0) = 0, \quad u_t(x,0) = xe^{-x^2},$$

  and using again the d'Alembert form of the solution we obtain that

  $$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} se^{-s^2} ds = -\frac{1}{4}e^{-s^2}|_{x-t}^{x+t} = \frac{1}{4}\left(e^{-(x-t)^2} - e^{-(x+t)^2}\right).$$

- 8.6 The given initial value problem is simply a superposition of the previous two initial value problems. Therefore, its solution is

$$u(x,t) = \frac{1}{2}[\sin(x-t) + \sin(x+t)] + \frac{1}{2}\int_{x-t}^{x+t} se^{-s^2}\,ds.$$

Integrating the second part of this formula we obtain that

$$u(x,t) = \frac{1}{2}[\sin(x-t) + \sin(x+t)] + \frac{1}{4}\left(e^{-(x-t)^2} - e^{-(x+t)^2}\right).$$

- 9.2 Using the d'Alembert form of the solution of the semi-infinite (one end fixed) boundary-initial value problem we obtain that

$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} se^{-s^2}\,ds = \frac{1}{4}\left[e^{-(x-t)^2} - e^{-(x+t)^2}\right] \quad \text{for} \quad x \geq t,$$

and

$$u(x,t) = \frac{1}{2}\int_{t-x}^{x+t} se^{-s^2}\,ds = \frac{1}{4}\left[e^{-(x-t)^2} - e^{-(x+t)^2}\right] \quad \text{for} \quad x < t$$

As both formulas are the same the solution can be represented as

$$u(x,t) = \frac{1}{4}\left[e^{-(x-t)^2} - e^{-(x+t)^2}\right] \quad \text{for} \quad 0 \leq x < \infty, \quad t \geq 0.$$

- 9.5 Using the same arguments as in the case of the semi-infinite string with the left boundary fixed, we look for the solution in the form

$$u(x,t) = F_1(x-ct) + G(x+ct), \quad when \quad x - ct < 0.$$

Thus,

$$u_x(x,t) = F_1'(x-ct) + G'(x+ct)$$

where "prime" denotes the ordinary derivative with respect the whole variable z equal either $x-ct$ or $x+ct$. Evaluating this derivative at the boundary $x=0$ we obtain that

$$F_1'(-ct) + G'(ct) = 0$$

which simply means that for any generic variable, say, s

$$F_1'(s) = -G'(-s).$$

Integrating this relation we obtain that

$$F_1(s) = G(-s).$$

Knowing from the original derivation of the d'Alembert solution that

$$G(s) = \frac{1}{2}f(s) + \frac{1}{2}\int_0^s g(u)du$$

and that the function F_1 must be evaluated for the variable $z = x - ct$, we obtain that

$$F_1(x - ct) = \frac{1}{2}f(ct - x) + \frac{1}{2c}\int_0^{ct-x} g(s)ds.$$