Homework 2
Solutions

- 4.5
  - (a) Given the Sine-Gordon equation

$$u_{tt} = u_{xx} - \sin x,$$

substitute the traveling wave solution $u(x,t) = f(x-ct)$ into the equation to find the required relation for the function $f(z)$, where $z = x - ct$. Using the Chain Rule, one can easily obtain that in order for $f(x-ct)$ to be a solution of the given equation the function f must satisfy the equation

$$c^2 f'' = f'' - \sin f$$

or equivalently

$$(1-c^2)f'' = \sin f.$$

  - (b) After multiplying the given differential equation for the function $f(z)$ by its derivative $f'(z)$ we obtain

$$(1-c^2)f''f' = f' \sin f.$$

This is a second-order ordinary differential equation. Integrating this equation once gives us

$$\frac{1}{2}(f'')^2(1-c^2) = -\cos f + A$$

which yields

$$(f')^2(1-c^2) = A - 2\cos f.$$

  - (c) Using the trigonometric identity

$$\cos f = 1 - 2\sin^2 \frac{f}{2}$$

and assuming that $A = 2$ one can re-write our equation in the form

$$(f')^2(1-c^2) = 4\sin^2 \frac{f}{2}.$$

To verify that $f(z) = 4\arctan\{\exp(\frac{x-ct}{\sqrt{1-c^2}})\}$ is a solution for $0 < c < 1$ observe that

$$\sin(\arctan f) = \frac{f}{\sqrt{1+f^2}}.$$

- 4.8 In order to determine if

$$u(x,t) = 4\arctan\{\exp(\frac{x-ct}{\sqrt{1-c^2}})\}$$

is a pulse, a wave front, or neither we should check the limits of the given function when $x \to \infty$ and $x \to -\infty$. You should be able to show that these are 2π and 0, respectively, proving that our solution is a *wave front.*

- 4.11 As shown in Example 4.10, a wave train solution to the Klein-Gordon equation propagates at speeds

$$c = \sqrt{a + \frac{ab}{\omega^2 - b}},$$

where a and b are positive. Note also that $\omega = \sqrt{ak^2 + b}$. This means that $\omega > \sqrt{b}$ as the wave number k can be arbitrarily small. It should now be easy to see that

$$\lim_{\omega \to \sqrt{b}} c = \infty$$

while

$$\lim_{\omega \to \infty} c = \sqrt{a}.$$

This proves that these wave trains can travel with speeds $\sqrt{a} < c < \infty$.

- 4.13 (a) Consider the differential equation

$$u_t + au_x = du_{xx}$$

where both constants are positive. To derive the dispersion relation, consider the solution

$$u(x,t) = e^{i(kx-wt)}.$$

Substituting this function into the differential equation and factoring by the common factor of $e^{i(kx-wt)}$, we obtain the following relation between w and k:

$$-iw + iak = -k^2 d.$$

Multiplying both sides by i and taking all the k dependent terms to the right, we obtain that

$$w = ak - k^2 di$$

which the dispersion relation for our differential equation. As w is a quadratic function of k, the said equation is dispersive.

- 5.2 Suppose $v(x,t)$ and $w(x,t)$ are two solutions to the equation $u_t + uu_x + u_{xxx} = 0$. Consider the function $u(x,t) = v(x,t) + w(x,t)$ and show that it is a solution to the given differential equation provided the product $v(x,t)w(x,t)$ is x independent.

  Substituting the function $u(x,t)$ into the left hand side of the differential equation we obtain that

  $$v_t + vv_v + v_{xxx} + w_t + ww_x + w_{xxx} + (vw_x + wv_x) = vw_x + wv_x$$

  as the functions v and w are the solutions of the original equation. Note now that $vw_x + wv_x = (vw)_x$. Thus, the right-hand side of the equation vanishes if the product vw is indeed x independent.

- 7.2 Note first that due to the assumption that $\rho(s) \approx \rho(x)$ for any s such that $x < s < x + \triangle x$, and that we only allow small deformations, the total mass of the segment $[x, x + \triangle x]$ is

  $$M = \int_x^{x+\triangle x} \rho(s)\sqrt{1 + (u_x)^2} ds \approx \int_x^{x+\triangle x} \rho(x) ds = \rho(x)\triangle x.$$

  This mean that the Newton's Second Law for this segment (compare with equation (7.4)) takes the form

  $$\rho(x)\triangle x u_{tt}(x,t) = T(x + \triangle x)u_x(x + \triangle x, t) - T(x)u_x(x,t).$$

  Dividing both sides by $\triangle x$ and taking the limit when $\triangle x \to 0$ we obtain the required equation

  $$\rho(x)u_{tt} = (T(x)u_x)_x.$$