Mth 322

Homework 1

Solutions

- 3.7 The function $u(x,t)$ is to satisfy the equation $u_t + u_x = 0$. This means that at every point x and time t $u_t = -u_x$. Thus, looking at a specific point x of the given graph (initial condition) and evaluating u_x at that point the equation tells you that the rate of change of the function u in time at that point is the negative of u_x. For example, if $u_x > 0$ at some point x then its value at that point must decrease in time. Knowing that the given profile must propagate rigidly through the medium (the given equation admits traveling waves) you should be able to determine that it must propagate to the right.

- 3.8
  - (a) The partial derivative $u_t(x,t)$ measures the speed, at the time t, at which the quantity u changes at the given point x in space. The second derivative $u_{tt}(x,t)$ measures the acceleration.
  - (b) It simply states that the acceleration is proportional to the concavity of the the string's shape.

- 3.11 Examples of first order nonlinear partial differential equations:

$$u_t + u^2 u_x = 0, \quad u_t + u^2 = 0.$$

- 3.12
  - (i) Order of equations: Equations (a) and (b) are of order one, the equation (f) is of order 3, all other equations are of order 2.
  - (ii) Equations (e) and (f) are *nonlinear*. The only *nonhomogeneous* equation is the equation (b).

- 4.4(a). We did this problem in class.

- 4.4(b). We should be looking for solutions $u(x,t) = f(x-ct)$ where both the function f and the speed c are unknown. Substituting it into the Klein-Gordon equation $u_{tt} = au_{xx} - bu$ we obtain the following ordinary differential equation for the function f:

$$c^2 f''(z) = af''(z) - bf(z),$$

where $f'(z) = \frac{df}{dz}$ and $z = x - ct$. This is a second-order, linear, ordinary differential equation with constant coefficients. Its characteristic equation is given by

$$(a - c^2)\lambda^2 - b = 0,$$

or equivalently

$$\lambda^2 = \frac{b}{a - c^2}.$$

Observe that $a \neq c^2$, that is, $c \neq \pm\sqrt{a}$ as otherwise the original equation implies that $f \equiv 0$, which is not considered a traveling wave. On the other hand, note that if $a > c^2$ the characteristic values are real while if $a < c^2$ they are imaginary. We shall consider these two cases separately. Namely,

  - (a) Let $a > c^2$, then

$$\lambda = \pm\sqrt{\frac{b}{a - c^2}}.$$

    Therefore

$$f(z) = A\exp(z\sqrt{\frac{b}{a-c^2}}) + B\exp(-z\sqrt{\frac{b}{a-c^2}}).$$

  - (b) Let $a < c^2$, then

$$\lambda = \pm i\sqrt{\frac{b}{c^2 - a}},$$

    and

$$f(z) = A\sin z\sqrt{\frac{b}{c^2-a}} + B\cos z\sqrt{\frac{b}{c^2-a}}.$$

    Substituting z by $x - ct$ we obtain the corresponding traveling wave solutions.