MTH 322
Review Problems

1. Consider the initial value problem

$$u_t + a(1-u)u_x = 0, \quad -\infty < x < \infty, \quad t > 0,$$

$$u(x,0) = \begin{cases} \frac{1}{2} & x < -1, \\ 0 & -1 < x < 0, \\ 1 & x > 0. \end{cases}$$

where a is a positive constant.

(a) Use the method of characteristics to show that the solution is a combination of a rarefaction wave and a shock wave.

(b) Find the speed of propagation of the shock wave part of this solution.

(c) Determine the time at which the rarefaction wave will start interacting with the shock wave and set up, but do not solve, the corresponding initial vale problem for the new shock wave.

2. Find the dispersion relation for the wave train solution of

$$u_t - u_x + u_{xxx} = 0.$$

Determine if the equation is dispersive. Remember that both the wave number and the circular frequency are assumed to be positive.

3. Find the solution to the following initial value problem:

$$u_t + (u^2 + 1)u_x = 0, \quad -\infty < x < \infty, \quad t > 0,$$

$$u(x,0) = \begin{cases} 0 & x < -1, \\ 1 & x > -1. \end{cases}$$

4. Show that the initial condition

$$u(x,0) = \begin{cases} 0 & x < 0, \\ 1 & x > 0. \end{cases}$$

will propagate as a shock wave solution of the differential equation

$$u_t - u^2 u_x = 0, \quad -\infty < x < \infty, \quad t > 0.$$

5. Use the characteristics to construct an xt-diagram representation for the d'Alembert solution of the semi-infinite problem

$$u_{tt} = u_{xx}, \quad 0 < x < \infty, \quad 0 < t \leq 2,$$

$$u(x,0) = \begin{cases} 0 & 0 < x < 1, \\ 1 & 1 < x < 2, \\ 0 & \text{otherwise}, \end{cases}$$

$$u_t(x,0) = 0, \quad u(0,t) = 0.$$

6. Find the traveling wave solutions of the differential equation

$$u_t + u_{xx} + u = 0, \quad -\infty < x < \infty, \quad t > 0,$$

and determine the speed(s) of propagation of these waves.

7. Consider the rarefaction wave solution from Problem 1. Show that it satisfies the entropy condition

$$\frac{u(x+h,t) - u(x,t)}{h} \leq \frac{E}{t}$$

for some positive constant E? Explain your answer.

8. Consider the following conservation law:

$$u_t - xt^2 u_x = 0, \quad -\infty < x < \infty, \quad t > 0.$$

   (a) Find its characteristic lines.

   (b) Find the solution to the initial condition

$$u(x,0) = \sin x.$$

   (c) Describe how the solution to the initial condition

$$u(x,0) = \begin{cases} 0 & x \leq 1, \\ 1 & x > 1, \end{cases}$$

   propagates.

9. Solve the initial-value problem

$$u_t + u_x\sqrt{u} = 0, \quad -\infty < x < \infty, \quad t > 0,$$

$$u(x,0) = \begin{cases} 0 & x < 0, \\ 1 & x > 0. \end{cases}$$

10. Consider the equation

$$u_t + 3u_{xxx} = 0.$$

   (a) Find all traveling wave solutions.

   (b) Show that the equation is dispersive.

11. Consider the initial-value problem:

$$u_t + 2uu_x = 0, \quad -\infty < x < \infty, \quad t > 0,$$

$$u(x,0) = \begin{cases} 2 & x \leq 0, \\ 1 & x > 0. \end{cases}$$

Use the *viscosity method* to find the solution.

12. Consider the conservation law

$$u_t + (u^2 - 1)u_x = 0.$$

The initial condition

$$u(x,0) = \begin{cases} 2 & x < 0, \\ 1 & 0 < x \leq 1, \\ 2 & x > 1. \end{cases}$$

propagates first, as easily determined by looking at the corresponding characteristics, as a combination of a shock-wave and a rarefaction-wave.

(a) Find the shock-wave part of this solution.

(b) Determine the time at which this initial shock will start interacting with the rarefaction part of the solution.

13. Find the d'Alembert solution to the initial-boundary value problem

$$u_{tt} = 4u_{xx}, \quad 0 < x < \infty, \quad t > 0,$$

$$u(x,0) = \sin x, \quad u_t(x,0) = \cos x,$$

$$u_x(0,t) = 0.$$