QUASIOPTIMALITY OF SOME SPECTRAL MIXED METHODS

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Abstract. In this paper we construct a sequence of projectors into certain polynomial spaces satisfying a commuting diagram property with norm bounds independent of the polynomial degree. Using the projectors we obtain quasioptimality of some spectral mixed methods, including the Raviart-Thomas method and mixed formulations of Maxwell equations. We also prove some discrete Friedrichs type inequalities involving curl.

1. Introduction

In this paper we show how one can use properties of certain regular right inverses of **grad**, **curl**, and div given by the classical Poincaré lemma in proving quasioptimality of some spectral mixed methods. We introduce the right inverses, establish their properties by elementary arguments, and discuss applications to the mixed methods.

For a large class of Galerkin methods, quasioptimality is immediate. Often the solution of a boundary value problem lies in a real Hilbert space V, its approximation u_n defined by a numerical method lies in a closed subspace $V_n \subset V$, and both are characterized by

$$a(u, v) = F(v), \qquad\qquad a(u_n, v_n) = F(v_n),$$

for all $v \in V$ and all $v_n \in V_n$. When $a(\cdot,\cdot)$ is a symmetric, coercive and continuous bilinear form on V and $F(\cdot)$ is a continuous functional on V, the approximation u_n is a projection of u in a norm equivalent to the norm on V (which we denote by $\|\cdot\|_V$), so there exists a constant $\mathcal{C}$ independent of V_n such that

$$\|u - u_n\|_V \le \mathcal{C} \inf_{v_n \in V_n} \|u - v_n\|_V,$$

i.e., the method is *quasioptimal*. Thus, the error analysis of the method immediately reduces to a question in approximation theory.

However, such a reduction is usually not so immediate for mixed systems. The Babuška-Brezzi theory of mixed systems provides two conditions under which one can obtain quasioptimality. Suppose W is a Hilbert space, W_n is a closed subspace of W, and $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are continuous bilinear forms on $V \times V$ and $V \times W$, respectively. Suppose further that the exact solution $u \in V, z \in W$ and the approximate solution $u_n \in V_n, z_n \in W_n$ solve the mixed systems

$$\begin{aligned} a(u, v) + b(v, z) &= F(v), \qquad & a(u_n, v_n) + b(v_n, z_n) &= F(v_n),\\ b(u, w) &= G(w), & b(u_n, w_n) &= G(w_n), \end{aligned}$$

for all $v \in V, w \in W, v_n \in V_n$, and $w_n \in W_n$, for some continuous functionals F and G. The conditions under which the above equations have a unique solution are well known [3]. It is shown in [3, Chapter II] that the quasioptimality estimate

$$\|u - u_n\|_V + \|z - z_n\|_{W/W^0} \le \mathcal{C}_{a,b,\alpha,\beta}\left(\inf_{v_n \in V_n} \|u - v_n\|_V + \inf_{w_n \in W_n} \|z - w_n\|_{W/W^0}\right) \tag{1.1}$$

holds provided the following two conditions hold:

$$\|w_n\|_{W/W^0} \leq \alpha \sup_{v_n \in V_n} \frac{b(v_n, w_n)}{\|v_n\|_V}, \qquad \text{for all } w_n \in W_n, \tag{1.2}$$

$$a(v,v) \geq \beta \, \|v\|_V^2, \qquad \text{for all } v \in V_n^0. \tag{1.3}$$

Here

$$V_n^0 = \{v \in V_n : b(v,w) = 0 \text{ for all } w \in W_n\},$$
$$W^0 = \{w \in W : b(v,w) = 0 \text{ for all } v \in V\}.$$

The constant $\mathcal{C}_{a,b,\alpha,\beta}$ in (1.1) depends only on α, β, and the norms of the bilinear forms a and b. Obviously, the estimate of (1.1) is interesting only if $\mathcal{C}_{a,b,\alpha,\beta}$ is independent of V_n and W_n. Therefore one needs to establish Conditions (1.2) and (1.3) with α and β independent of V_n and W_n. In the case of the spectral mixed methods we shall consider, the subspaces V_n and W_n will be polynomial spaces, so we will need to establish the above inequalities with constants independent of the polynomial degree.

The main theoretical device that helps us establish such inequalities are certain regular right inverses of the differential operators $\mathbf{grad}, \mathbf{curl}$, and div. These right inverses are constructed by means of explicit formulae involving certain line integrals. The integrals define appropriate vector and scalar potentials and are the same integrals that appear in the well known Poincaré lemma in differential geometry. The relevance of such potential mappings in the context of finite elements appears only to have been been noticed recently [10]. It was later utilized to prove optimal p interpolation estimates for triangular edge elements in [7]. The present work is motivated by the considerations in [7] and the results presented here are extensions of the two dimensional results there.

The essential idea of construction of the right inverses can best be revealed by first considering how one constructs a right inverse of the gradient operator. Given a smooth irrotational vector field $\boldsymbol{q}$ on $\mathbb{R}^3$, it is an elementary and well known result that one can construct a scalar potential ϕ such that $\mathbf{grad}\,\phi = \boldsymbol{q}$ by integration along lines. In other words, if we define the line integral of $\boldsymbol{q}$ from a fixed point $\boldsymbol{a}$ to $\boldsymbol{x}$, namely

$$G\boldsymbol{q}(\boldsymbol{x}) = \int_{\boldsymbol{a}}^{\boldsymbol{x}} \boldsymbol{q} \cdot \mathrm{d}\boldsymbol{t},$$

then by the fundamental theorem of calculus, $\mathbf{grad}\, G\boldsymbol{q} = \boldsymbol{q}$. Thus we have a right inverse of the gradient map. Similar ideas allow one to construct right inverses of divergence and curl in an elementary fashion.

We construct new projectors into spaces of certain polynomials using the right inverses. For example, we define a projector $\boldsymbol{\Pi}_p^Q$ into the Nédélec space which is well defined on all functions in $\boldsymbol{H}(\mathbf{curl}\,, \Omega)$. Note that in contrast, some of the standard projectors into the Nédélec space can be applied only to functions in $\boldsymbol{H}(\mathbf{curl}\,, \Omega)$ satisfying additional smoothness properties. Our main result concerning the projectors is a commuting diagram property (see Theorem 3.1), whose fundamental importance in the study of mixed methods has been clarified by many authors.

In Section 4, we consider the spectral Raviart-Thomas method for the Dirichlet problem on fairly general domains. We prove that the error of the method is equivalent to the best approximation error with constants of equivalence independent of the degree of polynomials. We prove this by establishing the appropriate Babuška-Brezzi inequality (1.2) with constant independent of polynomial degree p using one of our new projectors.

In Section 5, we prove some discrete Friedrichs inequalities. These will be seen in Section 6 to be useful in verifying the above mentioned condition (1.3) for a mixed method arising from discretization of Maxwell and Stokes equations. We prove their quasioptimality. In this application, we consider the case of homogeneous boundary conditions. To study this case, we construct a regular right inverse of curl that maps functions with zero normal traces to functions with zero tangential traces on the boundary. While we have explicit formulae for the inverses in the case of no boundary conditions, the construction of inverses that maintain boundary conditions is more subtle. In our construction, we make use of several recent results on characterization of traces [4, 5] of $\boldsymbol{H}(\mathbf{curl}\,,\Omega)$ on polyhedral boundary $\partial\Omega$, as well as an optimal polynomial extension operator [13].

2. Regular right inverses of $\mathbf{grad}, \mathbf{curl}$, and div

Let Ω be an open bounded set in $\mathbb{R}^3$ that is star shaped with respect to some point $\boldsymbol{a}$ in Ω or on its boundary. In the examples we have in mind, Ω will be a single "finite element", usually a simplex, a cube, or a prism. We assume that Ω has Lipschitz boundary. Since Ω is simply connected with connected boundary, it is well known that the following sequence is exact:

$$0 \longrightarrow H^1(\Omega)/\mathbb{R} \xrightarrow{\mathbf{grad}} \boldsymbol{H}(\mathbf{curl}\,,\Omega) \xrightarrow{\mathbf{curl}} \boldsymbol{H}(\mathrm{div},\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \longrightarrow 0. \tag{2.1}$$

In this section we will define bounded linear operators that traverse the sequence in the reverse order. Let $\mathcal{D}(\Omega)$ denote the set of infinitely differentiable functions that are compactly supported on Ω and let $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ denote the collection of vector functions that are restrictions to $\overline{\Omega}$ of infinitely differentiable compactly supported functions from $\mathbb{R}^3$ to $\mathbb{R}^3$. For $\psi \in \mathcal{D}(\Omega)$, $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$, and $\boldsymbol{q} \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$, define

$$\boldsymbol{D}\psi(\boldsymbol{x}) = \frac{2}{3}(\boldsymbol{x}-\boldsymbol{a})\int_0^1 t\,\psi(t^{2/3}(\boldsymbol{x}-\boldsymbol{a})+\boldsymbol{a})\,\mathrm{d}t, \tag{2.2}$$

$$\boldsymbol{K}\boldsymbol{v}(\boldsymbol{x}) = -2(\boldsymbol{x}-\boldsymbol{a})\times\int_0^1 t^3\boldsymbol{v}(t^2(\boldsymbol{x}-\boldsymbol{a})+\boldsymbol{a})\,\mathrm{d}t \tag{2.3}$$

$$G\boldsymbol{q}(\boldsymbol{x}) = (\boldsymbol{x}-\boldsymbol{a})\cdot\int_0^1 \boldsymbol{q}(t(\boldsymbol{x}-\boldsymbol{a})+\boldsymbol{a})\,\mathrm{d}t. \tag{2.4}$$

Note that a change of variable shows that $\boldsymbol{D}\psi(\boldsymbol{x}) = (\boldsymbol{x}-\boldsymbol{a})\int_0^1 t^2\,\psi(t(\boldsymbol{x}-\boldsymbol{a})+\boldsymbol{a})\,\mathrm{d}t$, and $\boldsymbol{K}\boldsymbol{v}(\boldsymbol{x}) = -(\boldsymbol{x}-\boldsymbol{a})\times\int_0^1 t\boldsymbol{v}(t(\boldsymbol{x}-\boldsymbol{a})+\boldsymbol{a})\,\mathrm{d}t$, but as we shall see, the expressions (2.2) and (2.3) are more convenient for estimation. We emphasize that these maps are identical to the maps of the classical Poincaré lemma (expressed usually in terms of differential forms as in [6, 10]).

As the estimates of our next theorem show, the maps defined by (2.2)–(2.4) extend as continuous linear operators between adjacent Sobolev spaces in (2.1). Moreover, the operators are such that div $\boldsymbol{D}$, $\mathbf{curl}\,\boldsymbol{K}$, and $\mathbf{grad}\,G$ are all identity maps on appropriate spaces. More precisely, letting

$$\boldsymbol{H}(\mathrm{div}\,0,\Omega) = \{\boldsymbol{v}\in\boldsymbol{H}(\mathrm{div},\Omega) : \mathrm{div}\,\boldsymbol{v} = 0\} \quad \text{and}$$
$$\boldsymbol{H}(\mathbf{curl}\,0,\Omega) = \{\boldsymbol{q}\in\boldsymbol{H}(\mathbf{curl}\,,\Omega) : \mathbf{curl}\,\boldsymbol{q} = 0\},$$

we have the following theorem.

Theorem 2.1. *The maps $\boldsymbol{D}$, $\boldsymbol{K}$, and G defined by (2.2)–(2.4) uniquely extend as continuous linear operators on the Sobolev space domains shown below:*

$$H^1(\Omega)/\mathbb{R} \xleftarrow{\;G\;} \boldsymbol{H}(\mathbf{curl}\,,\Omega) \xleftarrow{\;\boldsymbol{K}\;} \boldsymbol{H}(\mathrm{div},\Omega) \xleftarrow{\;\boldsymbol{D}\;} L^2(\Omega),$$

i.e., there are positive constants $\mathcal{K}_D$, $\mathcal{K}_K$ and $\mathcal{K}_G$ such that

$$\|\boldsymbol{D}\psi\|_{\boldsymbol{H}(\mathrm{div},\Omega)} \le \mathcal{K}_D\|\psi\|_{0,\Omega}, \qquad \text{for all } \psi \in L^2(\Omega), \tag{2.5}$$

$$\|\boldsymbol{K}\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl}\,,\Omega)} \le \mathcal{K}_K\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathrm{div},\Omega)} \qquad \text{for all } \boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},\Omega), \tag{2.6}$$

$$\|G\boldsymbol{q}\|_{H^1(\Omega)/\mathbb{R}} \le \mathcal{K}_G\|\boldsymbol{q}\|_{\boldsymbol{H}(\mathbf{curl}\,,\Omega)}, \qquad \text{for all } \boldsymbol{q} \in \boldsymbol{H}(\mathbf{curl}\,,\Omega). \tag{2.7}$$

Moreover, for all $\psi \in L^2(\Omega)$, $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}\,0,\Omega)$, and $\boldsymbol{q} \in \boldsymbol{H}(\mathbf{curl}\,0,\Omega)$,

$$\begin{aligned} \mathrm{div}\,\boldsymbol{D}\psi &= \psi, \\ \mathbf{curl}\,\boldsymbol{K}\boldsymbol{v} &= \boldsymbol{v}, \\ \mathbf{grad}\,G\boldsymbol{q} &= \boldsymbol{q}. \end{aligned}$$

The proof of this theorem will follow from some intermediate results we now present. We begin with three well known identities. They are usually proved using differential forms, but to emphasize their elementary nature, we prove one.

Proposition 2.1. *The following identities hold:*

$$\begin{aligned} \mathrm{div}\,\boldsymbol{D}\psi &= \psi && \text{for all } \psi \in \mathcal{D}(\Omega). \\ \mathbf{curl}\,\boldsymbol{K}\boldsymbol{v} &= \boldsymbol{v} - \boldsymbol{D}\,\mathrm{div}\,\boldsymbol{v} && \text{for all } \boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\overline{\Omega}). \\ \mathbf{grad}\,G\boldsymbol{q} &= \boldsymbol{q} - \boldsymbol{K}\mathbf{curl}\,\boldsymbol{q} && \text{for all } \boldsymbol{q} \in \boldsymbol{\mathcal{D}}(\overline{\Omega}). \end{aligned}$$

Proof. All three identities can be verified by elementary calculations. For example, setting $\boldsymbol{y}_t = t^2(\boldsymbol{x}-\boldsymbol{a})+\boldsymbol{a}$ and employing (temporarily) the summation convention together with the permutation symbol (ε_{ijk}), we have for any $\boldsymbol{v} \equiv (v_m) \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$,

$$\begin{aligned} -\frac{1}{2}[\mathbf{curl}\,(\boldsymbol{K}\boldsymbol{v})]_i &= \varepsilon_{ijk}\frac{\partial}{\partial x_j}\varepsilon_{klm}(x_l-a_l)\int_0^1 t^3 v_m(\boldsymbol{y}_t)\,\mathrm{d}t \\ &= (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})\left(\delta_{jl}\int_0^1 t^3 v_m(\boldsymbol{y}_t)\,\mathrm{d}t + (x_l-a_l)\int_0^1 t^3\frac{\partial}{\partial x_j}v_m(\boldsymbol{y}_t)\,\mathrm{d}t\right) \\ &= -2\int_0^1 t^3 v_i(\boldsymbol{y}_t)\,\mathrm{d}t - (x_j-a_j)\int_0^1 t^3\frac{\partial}{\partial x_j}v_i(\boldsymbol{y}_t)\,\mathrm{d}t + (x_i-a_i)\int_0^1 t^3\frac{\partial}{\partial x_j}v_j(\boldsymbol{y}_t)\,\mathrm{d}t, \end{aligned}$$

where δ denotes Kronecker delta. Since $dv_i/dt = 2t(\boldsymbol{x}-\boldsymbol{a})\cdot\mathbf{grad}_y\,v_i$, (where the subscript in $\mathbf{grad}_y$ indicates differentiation with respect to components of $\boldsymbol{y}_t$), we find that

$$-\frac{1}{2}[\mathbf{curl}\,(\boldsymbol{K}\boldsymbol{v})]_i = -2\int_0^1 t^3 v_i(\boldsymbol{y}_t)\,\mathrm{d}t - \frac{1}{2}\int_0^1 t^4\frac{dv_i}{dt}\,\mathrm{d}t + \frac{1}{2}[\boldsymbol{D}\,\mathrm{div}\,\boldsymbol{v}]_i.$$

Now an integration by parts shows that $\mathbf{curl}\,\boldsymbol{K}\boldsymbol{v} = \boldsymbol{v} - \boldsymbol{D}\,\mathrm{div}\,\boldsymbol{v}$. □

Lemma 2.1. *For all $\psi \in \mathcal{D}(\Omega)$, $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$, and $\boldsymbol{q} \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$, we have*

$$\|\boldsymbol{D}\psi\|_{0,\Omega} \le \mathcal{C}_D\|\psi\|_{0,\Omega}, \qquad \text{with } \mathcal{C}_D = 2h_\Omega/3, \tag{2.8}$$

$$\|\boldsymbol{K}\boldsymbol{v}\|_{0,\Omega} \le \mathcal{C}_K\|\boldsymbol{v}\|_{0,\Omega}, \qquad \text{with } \mathcal{C}_K = 2h_\Omega, \text{ and} \tag{2.9}$$

$$\|G\boldsymbol{q}\|_{H^1(\Omega)/\mathbb{R}} \le \mathcal{C}_G\|\boldsymbol{q}\|_{\boldsymbol{H}(\mathbf{curl}\,,\Omega)}, \tag{2.10}$$

with some constant $\mathcal{C}_G$ independent of $\boldsymbol{q}$. Here h_Ω denotes the diameter of Ω.

Proof. To prove the first inequality of the lemma, let $\boldsymbol{y}_t = t^{2/3}(\boldsymbol{x}-\boldsymbol{a})+\boldsymbol{a}$ and $\Omega_t = \{t^{2/3}(\boldsymbol{x}-\boldsymbol{a})+\boldsymbol{a} : \boldsymbol{x}\in\Omega\}$. Then,

$$\begin{aligned}\|\boldsymbol{D}\psi\|_{0,\Omega}^2 &= \int_\Omega \frac{4}{9}|\boldsymbol{x}-\boldsymbol{a}|^2\left(\int_0^1 t\psi(t^{2/3}(\boldsymbol{x}-\boldsymbol{a})+\boldsymbol{a})\,\mathrm{d}t\right)^2\mathrm{d}x\\ &\le \frac{4}{9}h_\Omega^2\int_0^1\int_\Omega t^2\,\psi(\boldsymbol{y}_t)^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \frac{4}{9}h_\Omega^2\int_0^1 t^2\int_{\Omega_t}\psi(\boldsymbol{y}_t)^2\,t^{-2}\mathrm{d}\boldsymbol{y}_t\,\mathrm{d}t\end{aligned}$$

Since $\Omega_t\subseteq\Omega$, we have

$$\begin{aligned}\|\boldsymbol{D}\psi\|_{0,\Omega}^2 &\le \frac{4}{9}h_\Omega^2\int_0^1\int_{\Omega_t}\psi(\boldsymbol{y}_t)^2\mathrm{d}\boldsymbol{y}_t\,\mathrm{d}t\\ &\le \frac{4}{9}h_\Omega^2\int_0^1\|\psi\|_{0,\Omega}^2\,\mathrm{d}t = \frac{4}{9}h_\Omega^2\,\|\psi\|_{0,\Omega}^2.\end{aligned}$$

To prove (2.9), let $\boldsymbol{y}_t$ now denote $t^2(\boldsymbol{x}-\boldsymbol{a})+\boldsymbol{a}$ and $\Omega_t=\{t^2(\boldsymbol{x}-\boldsymbol{a})+\boldsymbol{a} : \boldsymbol{x}\in\Omega\}$. Then

$$\begin{aligned}\|\boldsymbol{K}\boldsymbol{v}\|_{0,\Omega}^2 &\le \int_\Omega 4|\boldsymbol{x}-\boldsymbol{a}|^2\int_0^1 t^6|\boldsymbol{v}(t^2(\boldsymbol{x}-\boldsymbol{a})+\boldsymbol{a})|^2\,\mathrm{d}t\,\mathrm{d}x\\ &\le 4h_\Omega^2\int_0^1 t^6\int_\Omega|\boldsymbol{v}(\boldsymbol{y}_t)|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\ &= 4h_\Omega^2\int_0^1 t^6\int_{\Omega_t}|\boldsymbol{v}(\boldsymbol{y}_t)|^2t^{-6}\,\mathrm{d}\boldsymbol{y}_t\,\mathrm{d}t = 4h_\Omega^2\int_0^1\|\boldsymbol{v}\|_{0,\Omega_t}^2\,\mathrm{d}t\\ &\le 4h_\Omega^2\|\boldsymbol{v}\|_{0,\Omega}^2.\end{aligned}$$

Finally, to prove (2.10), we use Friedrichs inequality, which asserts the existence of a constant $\mathcal{C}_{\mathrm{Fr}}>0$, depending on Ω, such that

$$\|\phi\|_{L^2(\Omega)/\mathbb{R}} = \inf_{c\in\mathbb{R}}\|\phi-c\|_{0,\Omega}\le\mathcal{C}_{\mathrm{Fr}}\|\,\mathbf{grad}\,\phi\|_{0,\Omega},\quad\text{for all }\phi\in H^1(\Omega). \tag{2.11}$$

Together with the third identity of Proposition 2.1 and (2.9), this implies that

$$\begin{aligned}\|G\boldsymbol{q}\|_{L^2(\Omega)/\mathbb{R}} &\le \mathcal{C}_{\mathrm{Fr}}\|\,\mathbf{grad}\,G\boldsymbol{q}\|_{0,\Omega} = \mathcal{C}_{\mathrm{Fr}}\|\boldsymbol{q}-\boldsymbol{K}\,\mathbf{curl}\,\boldsymbol{q}\|_{0,\Omega}\\ &\le \mathcal{C}_{\mathrm{Fr}}(\|\boldsymbol{q}\|_{0,\Omega}+2h_\Omega\|\mathbf{curl}\,\boldsymbol{q}\|_{0,\Omega}),\end{aligned}$$

from which (2.10) follows. □

Remark 2.1. Although Theorem 2.1 only asserted the continuity of $\boldsymbol{K}:\boldsymbol{H}(\mathrm{div},\Omega)\mapsto\boldsymbol{H}(\mathbf{curl}\,,\Omega)$, note that Lemma 2.1 gives a stronger result: the boundedness of $\boldsymbol{K}$ on $L^2(\Omega)^3$. The map $\boldsymbol{D}$ is also L^2-bounded. However, the map G is not well defined on all $L^2(\Omega)^3$. Indeed, on any domain $\Omega\subset\mathbb{R}^3$ containing the origin, the function $\boldsymbol{q}(\boldsymbol{x})=\boldsymbol{x}/|\boldsymbol{x}|^2$ is in $L^2(\Omega)^3$, but the integral in the definition of $G\boldsymbol{q}$ does not exist.

Proof of Theorem 2.1. By Lemma 2.1 and Proposition 2.1,

$$\begin{aligned}\|\boldsymbol{D}\psi\|_{\boldsymbol{H}(\mathrm{div},\Omega)}^2 &= \|\,\mathrm{div}\,\boldsymbol{D}\psi\|_{0,\Omega}^2+\|\boldsymbol{D}\psi\|_{0,\Omega}^2\le(1+\mathcal{C}_D^2)\|\psi\|_{0,\Omega}^2,\\ \|\boldsymbol{K}\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl}\,,\Omega)}^2 &= \|\boldsymbol{v}-\boldsymbol{D}\,\mathrm{div}\,\boldsymbol{v}\|_{0,\Omega}^2+\|\boldsymbol{K}\boldsymbol{v}\|_{0,\Omega}^2\\ &\le 2\|\boldsymbol{v}\|_{0,\Omega}^2+2\mathcal{C}_D^2\|\,\mathrm{div}\,\boldsymbol{v}\|_{0,\Omega}+\mathcal{C}_K^2\|\boldsymbol{v}\|_{0,\Omega}^2,\end{aligned}$$

for all $\psi \in \mathcal{D}(\Omega)$ and $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$. An analogous estimate for G is also given by Lemma 2.1. Now, it follows that the maps $\boldsymbol{D}$, $\boldsymbol{K}$, and G are well defined on the Sobolev spaces asserted by the theorem because of the density of $\mathcal{D}(\Omega)$ in $L^2(\Omega)$ and the density of $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ in $\boldsymbol{H}(\mathbf{curl}\,,\Omega)$ and $\boldsymbol{H}(\mathrm{div},\Omega)$ proved in [9, Chapter I]. Inequalities (2.5)–(2.7) are thus proved.

It remains to show the equalities of the theorem. To prove that $\boldsymbol{K}$ is a right inverse of $\mathbf{curl}$, let $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},\Omega)$ and choose a sequence $\boldsymbol{v}_n \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$ that converges to $\boldsymbol{v}$ in $\boldsymbol{H}(\mathrm{div},\Omega)$–norm. By Proposition 2.1,

$$\mathbf{curl}\,\boldsymbol{K}\boldsymbol{v}_n = \boldsymbol{v}_n - \boldsymbol{D}\,\mathrm{div}\,\boldsymbol{v}_n.$$

The right hand side of this equality converges to $\boldsymbol{v} - \boldsymbol{D}\,\mathrm{div}\,\boldsymbol{v}$ because of (2.5), while the left hand side converges to $\mathbf{curl}\,\boldsymbol{K}\boldsymbol{v}$ because of (2.6). Thus $\mathbf{curl}\,\boldsymbol{K}\boldsymbol{v} = \boldsymbol{v}$ for all $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}\,0,\Omega)$. Proofs of the other identities are similar. □

Remark 2.2. Observe that the divergence and gradient maps remain at the tail and front end of (2.1), respectively, in any space dimension. However, there are more than one intermediate members in the sequence when $N > 3$. For simplicity, we shall not discuss construction of inverses for analogues of curl in more than three dimensions. But the definitions of right inverses of divergence and gradient have natural extensions to N-dimensional vector fields. Indeed, if $\boldsymbol{D}$ is defined in N-dimensions by

$$\boldsymbol{D}\psi(\boldsymbol{x}) = \frac{2}{N}(\boldsymbol{x}-\boldsymbol{a})\int_0^1 t\,\psi(t^{2/N}(\boldsymbol{x}-\boldsymbol{a})+\boldsymbol{a})\,\mathrm{d}t,$$

then $\mathrm{div}\,\boldsymbol{D}\psi = \psi$ for all $\psi \in L^2(\Omega)$ and (2.8) holds provided we redefine

$$\mathcal{C}_D = \frac{2}{N}h_\Omega.$$

Similarly, the equality $\mathbf{grad}\,G\boldsymbol{q} = \boldsymbol{q}$ continues to hold in N dimensions provided we choose $\boldsymbol{q}$ in

$$\{\boldsymbol{q} \equiv (q_i) \in L^2(\Omega)^N : \partial q_j/\partial x_i - \partial q_i/\partial x_j = 0 \text{ for all } i,j = 1,2,\ldots,N, i \neq j\},$$

instead of in $\boldsymbol{H}(\mathbf{curl}\,0,\Omega)$.

3. Commuting projections

In this section we define projectors into certain polynomial spaces and establish a commuting diagram property. The polynomial spaces we shall consider are the Raviart-Thomas [15] and Nédélec spaces [14]. Let P_p denote the set of all polynomials of degree at most p and $\boldsymbol{P}_p$ denote the set of all vector polynomials whose three components are in P_p. Define $\boldsymbol{R}_p = \{\boldsymbol{r} \in \boldsymbol{P}_{p+1} : \boldsymbol{r} = \boldsymbol{x}p + \boldsymbol{q}$ for some scalar polynomial $p \in P_p$ and some vector polynomial $\boldsymbol{q} \in \boldsymbol{P}_p\}$ and $\boldsymbol{Q}_p = \{\boldsymbol{q} \in P_{p+1} : \boldsymbol{q} = \widetilde{\boldsymbol{q}} + \boldsymbol{q}_p$ for some homogeneous vector polynomial $\widetilde{\boldsymbol{q}}$ of degree $p+1$ such that $\widetilde{\boldsymbol{q}}\cdot\boldsymbol{x} = 0$ and some $\boldsymbol{q}_p \in \boldsymbol{P}_p\}$. These spaces are well known to possess the exact sequence property

$$0 \longrightarrow P_{p+1}/\mathbb{R} \xrightarrow{\mathbf{grad}} \boldsymbol{Q}_p \xrightarrow{\mathbf{curl}} \boldsymbol{R}_p \xrightarrow{\mathrm{div}} P_p \longrightarrow 0, \tag{3.1}$$

in analogy with (2.1).

Let Π_p denote the $L^2(\Omega)$ orthogonal projection into P_p, $\boldsymbol{\Pi}_p^{R0}$ denote the $L^2(\Omega)^3$ orthogonal projection into

$$\boldsymbol{R}_p^0 = \{\boldsymbol{r} \in \boldsymbol{R}_p : \mathrm{div}\,\boldsymbol{r} = 0\}, \tag{3.2}$$

$\boldsymbol{\Pi}_p^{Q0}$ denote the $L^2(\Omega)^3$ orthogonal projection into

$$\boldsymbol{Q}_p^0 = \{\boldsymbol{q} \in \boldsymbol{Q}_p : \mathbf{curl}\,\boldsymbol{q} = 0\},$$

and $\Pi^{W0} w = \text{meas}(\Omega)^{-1} \int_\Omega w \, \mathrm{d}x$. Define

$$\boldsymbol{\Pi}_p^R \boldsymbol{v} = \boldsymbol{\Pi}_p^{R0}\boldsymbol{v} + (\boldsymbol{I} - \boldsymbol{\Pi}_p^{R0})\boldsymbol{D}(\Pi_p \operatorname{div} \boldsymbol{v}), \tag{3.3}$$

$$\boldsymbol{\Pi}_p^Q \boldsymbol{q} = \boldsymbol{\Pi}_p^{Q0}\boldsymbol{q} + (\boldsymbol{I} - \boldsymbol{\Pi}_p^{Q0})\boldsymbol{K}(\boldsymbol{\Pi}_p^{R0}\mathbf{curl}\,\boldsymbol{q}), \tag{3.4}$$

$$\Pi_p^W w = \Pi^{W0} w + (I - \Pi^{W0}) G(\boldsymbol{\Pi}_p^{Q0}\,\mathbf{grad}\, w). \tag{3.5}$$

Our main result concerning these projectors is Theorem 3.1 given later in this section. But first, let us prove that the above operators are indeed projectors into the polynomial spaces introduced above. This is immediately seen from the following result.

Proposition 3.1.

(1) *For all* $\boldsymbol{r} \in \boldsymbol{H}(\operatorname{div}, \Omega)$, $\boldsymbol{\Pi}_p^R \boldsymbol{r} \in \boldsymbol{R}_p$. *Moreover, if* $\boldsymbol{r} \in \boldsymbol{R}_p$ *then* $\boldsymbol{\Pi}_p^R \boldsymbol{r} = \boldsymbol{r}$.
(2) *For all* $\boldsymbol{q} \in \boldsymbol{H}(\mathbf{curl}\,, \Omega)$, $\boldsymbol{\Pi}_p^Q \boldsymbol{q} \in \boldsymbol{Q}_p$. *Moreover, if* $\boldsymbol{q} \in \boldsymbol{Q}_p$ *then* $\boldsymbol{\Pi}_p^Q \boldsymbol{q} = \boldsymbol{q}$.
(3) *For all* $w \in H^1(\Omega)$, $\Pi_p^W w \in P_{p+1}$. *Moreover, if* $w \in P_{p+1}$ *then* $\Pi_p^W w = w$.

Proof. To prove the first statement, note that whenever $\psi \in P_p$ the integral

$$\int_0^1 t\, \psi(t^{2/3}(\boldsymbol{x} - \boldsymbol{a}) + \boldsymbol{a}) \, \mathrm{d}t$$

also yields a function in P_p. Since $\Pi_p \operatorname{div} \boldsymbol{v} \in P_p$, we have that $\boldsymbol{D}\Pi_p \operatorname{div} \boldsymbol{v} \in \boldsymbol{R}_p$ for all $\boldsymbol{v} \in \boldsymbol{H}(\operatorname{div}, \Omega)$, so it follows that $\boldsymbol{\Pi}_p^R \boldsymbol{v} \in \boldsymbol{R}_p$. Now consider an $\boldsymbol{r} \in \boldsymbol{R}_p$. We need to show that $\boldsymbol{\Pi}_p^R \boldsymbol{r} \in \boldsymbol{R}_p$. Decompose $\boldsymbol{r}$ as

$$\boldsymbol{r} = \boldsymbol{\Pi}_p^{R0}\boldsymbol{r} + (\boldsymbol{I} - \boldsymbol{\Pi}_p^{R0})\boldsymbol{r}.$$

Since $\operatorname{div} \boldsymbol{r}$ is in P_p,

$$\boldsymbol{\Pi}_p^R \boldsymbol{r} = \boldsymbol{\Pi}_p^{R0}\boldsymbol{r} + (\boldsymbol{I} - \boldsymbol{\Pi}_p^{R0})\boldsymbol{D} \operatorname{div} \boldsymbol{r}.$$

Therefore,

$$\boldsymbol{r} - \boldsymbol{\Pi}_p^R \boldsymbol{r} = (\boldsymbol{I} - \boldsymbol{\Pi}_p^{R0})(\boldsymbol{r} - \boldsymbol{D} \operatorname{div} \boldsymbol{r}). \tag{3.6}$$

By Theorem 2.1, $\operatorname{div}(\boldsymbol{I} - \boldsymbol{\Pi}_p^{R0})(\boldsymbol{r} - \boldsymbol{D} \operatorname{div} \boldsymbol{r}) = \operatorname{div}(\boldsymbol{r} - \boldsymbol{D} \operatorname{div} \boldsymbol{r}) = 0$. Hence $\boldsymbol{r} - \boldsymbol{\Pi}_p^R \boldsymbol{r}$ is simultaneously in the range of $\boldsymbol{\Pi}_p^{R0}$ and $\boldsymbol{I} - \boldsymbol{\Pi}_p^{R0}$, so must vanish.

To prove the statement about the operator $\boldsymbol{\Pi}_p^Q$, we again note that since $\boldsymbol{v} \equiv \boldsymbol{\Pi}_p^R \mathbf{curl}\,\boldsymbol{q} \in \boldsymbol{R}_p$, the integral

$$\boldsymbol{k} = \int_0^1 t^3 \boldsymbol{v}(t^2(\boldsymbol{x} - \boldsymbol{a}) + \boldsymbol{a}) \, \mathrm{d}t$$

is a polynomial in $\boldsymbol{R}_p$. Let $\widetilde{q}_p$ be the homogeneous polynomial of degree p such that $\boldsymbol{k} = \boldsymbol{x}\widetilde{q}_p + \boldsymbol{q}_p$ with $\boldsymbol{q}_p \in \boldsymbol{P}_p$. Then

$$\boldsymbol{K}\boldsymbol{v} = -2(\boldsymbol{x} - \boldsymbol{a}) \times \boldsymbol{k} = -2(\boldsymbol{x} \times \boldsymbol{q}_p + 2\boldsymbol{a} \times \boldsymbol{x}\widetilde{q}_p) + 2\boldsymbol{a} \times \boldsymbol{q}_p.$$

Since $\boldsymbol{x} \cdot (\boldsymbol{x} \times \boldsymbol{q}_p + 2\boldsymbol{a} \times \boldsymbol{x}\widetilde{q}_p) = 0$ and $2\boldsymbol{a} \times \boldsymbol{q}_p \in \boldsymbol{P}_p$ we find that $\boldsymbol{K}\boldsymbol{q} \in \boldsymbol{Q}_p$, so $\boldsymbol{\Pi}_p^Q \boldsymbol{v}$ is in $\boldsymbol{Q}_p$. To prove that $\boldsymbol{\Pi}_p^Q \boldsymbol{q} = \boldsymbol{q}$ for all $\boldsymbol{q} \in \boldsymbol{Q}_p$ we proceed as in the previous case: Since

$$\boldsymbol{q} - \boldsymbol{\Pi}_p^Q \boldsymbol{q} = (\boldsymbol{I} - \boldsymbol{\Pi}_p^{Q0})(\boldsymbol{q} - \boldsymbol{K}\mathbf{curl}\,\boldsymbol{q}),$$

we find from Theorem 2.1 that $\mathbf{curl}\,(\boldsymbol{q}-\boldsymbol{\Pi}_p^Q\boldsymbol{q}) = \mathbf{curl}\,(\boldsymbol{q}-\boldsymbol{K}\mathbf{curl}\,\boldsymbol{q}) = 0$. Therefore $\boldsymbol{q}-\boldsymbol{\Pi}_p^Q\boldsymbol{q}$ is in the range of $\boldsymbol{\Pi}_p^{Q0}$ and at the same time in the range of $\boldsymbol{I}-\boldsymbol{\Pi}_p^{Q0}$. Hence $\boldsymbol{q}-\boldsymbol{\Pi}_p^Q\boldsymbol{q} = 0$.

The final statement of the proposition is proved similarly. □

Remark 3.1. It is possible to compute the projections defined in (3.3)–(3.5) without using the right inverse maps. Indeed, $\boldsymbol{\Pi}_p^R\boldsymbol{v}$ equals the unique function $\boldsymbol{\pi}$ in $\boldsymbol{R}_p$ satisfying

$$\begin{aligned} (\boldsymbol{\pi},\boldsymbol{r}_0) &= (\boldsymbol{v},\boldsymbol{r}_0) && \text{for all } \boldsymbol{r}_0\in\boldsymbol{R}_p^0, \text{ and}\\ (\operatorname{div}\boldsymbol{\pi},\operatorname{div}\boldsymbol{r}) &= (\operatorname{div}\boldsymbol{v},\operatorname{div}\boldsymbol{r}) && \text{for all } \boldsymbol{r}\in\boldsymbol{R}_p, \end{aligned}$$

where $(\cdot,\cdot)$ denote the $L^2(\Omega)$ or $L^2(\Omega)^3$ innerproduct. To see this it is enough to observe that the second equation above implies that

$$\operatorname{div}(\boldsymbol{\pi}-\boldsymbol{\Pi}_p^R\boldsymbol{v}) = 0$$

while the first equation implies that $\boldsymbol{\pi}-\boldsymbol{\Pi}_p^R\boldsymbol{v}$ is orthogonal to $\boldsymbol{R}_p^0$. Thus, $\boldsymbol{\pi}-\boldsymbol{\Pi}_p^R\boldsymbol{v}$ is in $\boldsymbol{R}_p^0$ as well as its orthogonal complement, so vanishes. In the same way, $\boldsymbol{\Pi}_p^Q\boldsymbol{q}$ is characterized as the unique function in $\boldsymbol{Q}_p$ satisfying

$$\begin{aligned} (\boldsymbol{\Pi}_p^Q\boldsymbol{q},\boldsymbol{z}_0) &= (\boldsymbol{q},\boldsymbol{z}_0) && \text{for all } \boldsymbol{z}_0\in\boldsymbol{Q}_p^0, \text{ and}\\ (\mathbf{curl}\,\boldsymbol{\Pi}_p^Q\boldsymbol{q},\mathbf{curl}\,\boldsymbol{z}) &= (\mathbf{curl}\,\boldsymbol{q},\mathbf{curl}\,\boldsymbol{z}) && \text{for all } \boldsymbol{z}\in\boldsymbol{Q}_p. \end{aligned}$$

A similar characterization holds for Π_p^W as well.

Theorem 3.1. *The following diagram commutes:*

$$\begin{array}{ccccccccc} H^1(\Omega)/\mathbb{R} & \xrightarrow{\mathbf{grad}} & \boldsymbol{H}(\mathbf{curl},\Omega) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}(\operatorname{div},\Omega) & \xrightarrow{\operatorname{div}} & L^2(\Omega) & \longrightarrow & 0\\ \big\downarrow \Pi_p^W & & \big\downarrow \boldsymbol{\Pi}_p^Q & & \big\downarrow \boldsymbol{\Pi}_p^R & & \big\downarrow \Pi_p & & \\ P_{p+1}/\mathbb{R} & \xrightarrow{\mathbf{grad}} & \boldsymbol{Q}_p & \xrightarrow{\mathbf{curl}} & \boldsymbol{R}_p & \xrightarrow{\operatorname{div}} & P_p & \longrightarrow & 0 \end{array} \tag{3.7}$$

Moreover, the norms of all the projectors above are bounded independently of p:

$$\begin{aligned} \|\boldsymbol{\Pi}_p^R\boldsymbol{v}\|^2_{\boldsymbol{H}(\operatorname{div},\Omega)} &\le (1+\mathcal{C}_D^2)\|\boldsymbol{v}\|^2_{\boldsymbol{H}(\operatorname{div},\Omega)} && \textit{for all } \boldsymbol{v}\in\boldsymbol{H}(\operatorname{div},\Omega),\\ \|\boldsymbol{\Pi}_p^Q\boldsymbol{q}\|^2_{\boldsymbol{H}(\mathbf{curl},\Omega)} &\le (1+\mathcal{C}_K^2)\|\boldsymbol{q}\|^2_{\boldsymbol{H}(\mathbf{curl},\Omega)} && \textit{for all } \boldsymbol{q}\in\boldsymbol{H}(\mathbf{curl},\Omega),\\ \|\Pi_p^W w\|^2_{H^1(\Omega)} &\le (1+\mathcal{C}_G^2)\|w\|^2_{H^1(\Omega)} && \textit{for all } w\in H^1(\Omega). \end{aligned}$$

Proof. To prove that

$$\operatorname{div}\boldsymbol{\Pi}_p^R\boldsymbol{q} = \Pi_p\operatorname{div}\boldsymbol{q},$$

we use Theorem 2.1:

$$\begin{aligned} \operatorname{div}(\boldsymbol{\Pi}_p^R\boldsymbol{v}) &= \operatorname{div}(\boldsymbol{\Pi}_p^{R0}\boldsymbol{v}) + \operatorname{div}(\boldsymbol{I}-\boldsymbol{\Pi}_p^{R0})\boldsymbol{D}\Pi_p\operatorname{div}\boldsymbol{v}\\ &= \operatorname{div}\boldsymbol{D}\Pi_p\operatorname{div}\boldsymbol{v} = \Pi_p\operatorname{div}\boldsymbol{v}. \end{aligned}$$

Proofs of

$$\mathbf{curl}\,\boldsymbol{\Pi}_p^Q\boldsymbol{q} = \boldsymbol{\Pi}_p^R\mathbf{curl}\,\boldsymbol{q} \quad\text{and}\quad \mathbf{grad}\,\Pi_p^W\phi = \boldsymbol{\Pi}_p^Q\,\mathbf{grad}\,\phi$$

proceed similarly using the other identities of Theorem 2.1.

To prove the norm bound on $\boldsymbol{\Pi}_p^R$ we use Theorem 2.1 again. Since norms of orthogonal projectors equal one,

$$\begin{aligned}\|\boldsymbol{\Pi}_p^R \boldsymbol{v}\|_{0,\Omega}^2 &= \|\boldsymbol{\Pi}_p^{R0}\boldsymbol{v}\|_{0,\Omega}^2 + \|(\boldsymbol{I}-\boldsymbol{\Pi}_p^{R0})\boldsymbol{D}\Pi_p \operatorname{div}\boldsymbol{v}\|_{0,\Omega}^2 \\ &\le \|\boldsymbol{v}\|_{0,\Omega}^2 + \|\boldsymbol{D}\Pi_p \operatorname{div}\boldsymbol{v}\|_{0,\Omega}^2 \\ &\le \|\boldsymbol{v}\|_{0,\Omega}^2 + \mathcal{C}_D^2\|\operatorname{div}\boldsymbol{v}\|_{0,\Omega}^2, \quad \text{and} \\ \|\operatorname{div}\boldsymbol{\Pi}_p^R\boldsymbol{v}\|_{0,\Omega}^2 &= \|\Pi_p \operatorname{div}\boldsymbol{v}\|_{0,\Omega}^2 \le \|\operatorname{div}\boldsymbol{v}\|_{0,\Omega}^2.\end{aligned}$$

This proves that

$$\|\boldsymbol{\Pi}_p^R\boldsymbol{v}\|_{\boldsymbol{H}(\operatorname{div},\Omega)}^2 \le (1+\mathcal{C}_D^2)\|\boldsymbol{v}\|_{\boldsymbol{H}(\operatorname{div},\Omega)}^2.$$

The remaining estimates are proved similarly. □

Remark 3.2. Observe that the norm bounds of Theorem 3.1 imply quasioptimality of the projectors, i.e., the error in projection is bounded by a constant times the best approximation error. For example, for any $\boldsymbol{r}\in\boldsymbol{R}_p$,

$$\begin{aligned}\|\boldsymbol{v}-\boldsymbol{\Pi}_p^R\boldsymbol{v}\|_{\boldsymbol{H}(\operatorname{div},K)} &= \|(\boldsymbol{v}-\boldsymbol{r})-\boldsymbol{\Pi}_p^R(\boldsymbol{v}-\boldsymbol{r})\|_{\boldsymbol{H}(\operatorname{div},K)} \\ &\le (1+\|\boldsymbol{\Pi}_p^R\|_{\boldsymbol{H}(\operatorname{div},K)})\|\boldsymbol{v}-\boldsymbol{r}\|_{\boldsymbol{H}(\operatorname{div},K)} \\ &\le \mathcal{C}_1\|\boldsymbol{v}-\boldsymbol{r}\|_{\boldsymbol{H}(\operatorname{div},K)},\end{aligned}$$

where $\mathcal{C}_1 = (1+(1+\mathcal{C}_D^2)^{1/2})$. Thus,

$$\inf_{\boldsymbol{r}_p\in\boldsymbol{R}_p}\|\boldsymbol{v}-\boldsymbol{r}_p\|_{\boldsymbol{H}(\operatorname{div},K)} \le \|\boldsymbol{v}-\boldsymbol{\Pi}_p^R\boldsymbol{v}\|_{\boldsymbol{H}(\operatorname{div},K)} \le \mathcal{C}_1 \inf_{\boldsymbol{r}_p\in\boldsymbol{R}_p}\|\boldsymbol{v}-\boldsymbol{r}_p\|_{\boldsymbol{H}(\operatorname{div},K)}.$$

Similar equivalences hold for the other projectors as well. Thus, to obtain p-error estimates for these projectors, it suffices to estimate the best approximation error as a function of p.

Remark 3.3. We used two critical properties of the right inverses in obtaining the results of this section, namely (i) their continuity as given by the bounds (2.5)–(2.7) and (ii) the fact that they map a polynomial space in (3.1) into its adjacent left one:

$$P_{p+1}/\mathbb{R} \xleftarrow{\;G\;} \boldsymbol{Q}_p \xleftarrow{\;\boldsymbol{K}\;} \boldsymbol{R}_p \xleftarrow{\;\boldsymbol{D}\;} P_p. \tag{3.8}$$

The projectors defined by (3.3)–(3.5) remain unchanged if, in their definition, we replace our right inverses by any other right inverses satisfying the above mentioned two properties (see also Remark 3.1).

Remark 3.4. Right inverses of the divergence map have been constructed by other methods earlier [1, 16]. These constructions satisfy one or the other of the two properties mentioned in Remark 3.3, but not both in general.

Remark 3.5. It is possible to extend our results to *tensor product polynomials*. Let $P^{\boxplus}_{l,m,n}$ denote the set of polynomials in $\boldsymbol{x}\equiv(x_1,x_2,x_3)$ that are of degree at most l, m, and n in x_1, x_2, and x_3, respectively. Let $\boldsymbol{Q}_p^{\boxplus} = P^{\boxplus}_{p-1,p,p}\times P^{\boxplus}_{p,p-1,p}\times P^{\boxplus}_{p,p,p-1}$, $\boldsymbol{R}_p^{\boxplus} = P^{\boxplus}_{p,p-1,p-1}\times P^{\boxplus}_{p-1,p,p-1}\times P^{\boxplus}_{p-1,p-1,p}$. Then analogous to (3.1) and (2.1), the sequence

$$0 \longrightarrow P^{\boxplus}_{p,p,p}/\mathbb{R} \xrightarrow{\;\mathbf{grad}\;} \boldsymbol{Q}_p^{\boxplus} \xrightarrow{\;\mathbf{curl}\;} \boldsymbol{R}_p^{\boxplus} \xrightarrow{\;\operatorname{div}\;} P^{\boxplus}_{p-1,p-1,p-1} \longrightarrow 0$$

is exact. It is an easy matter to verify that the right inverses map a polynomial space in the above sequence into its adjacent left one:

$$P^{\boxplus}_{p,p,p}/\mathbb{R} \xleftarrow{\;G\;} \boldsymbol{Q}_p^{\boxplus} \xleftarrow{\;\boldsymbol{K}\;} \boldsymbol{R}_p^{\boxplus} \xleftarrow{\;\boldsymbol{D}\;} P^{\boxplus}_{p-1,p-1,p-1}.$$

Consequently, if the L^2-orthogonal projections in the definitions (3.3)–(3.5) are replaced by L^2-orthogonal projections into the tensor product spaces, then the norm estimates of Theorem 3.1 hold without change and diagram (3.7), after substitution with the tensor product spaces, commutes. It is also possible to extend such results to the sequence starting with $P^{\boxplus}_{p_1,p_2,p_3}/\mathbb{R}$ where the degrees p_i in different directions are not necessarily equal.

4. Application to the Raviart-Thomas mixed method

In this section we prove the quasioptimality of the spectral Raviart-Thomas mixed method for the Dirichlet problem. We will consider general Lipschitz domains Ω in $\mathbb{R}^N$ which are merely assumed to be star-shaped with respect to $\boldsymbol{a} \in \Omega$. In practical computations, the class of domains of interest is generally much smaller as one would need to get a computable basis convenient for computation and approximate the integrals over Ω required by the method in an efficient manner.

By means of the interpolant $\boldsymbol{\Pi}_p^R$, we will show that the problem of error estimation of the p-Raviart-Thomas method reduces to a problem of best approximation. As indicated in Remark 2.2, the definition of $\boldsymbol{\Pi}_p^R$ and its properties extends verbatim to N dimensions (with the exception of the change in value of $\mathcal{C}_D$).

Let $\boldsymbol{q}$ and u solve the following Dirichlet problem on Ω:

$$\boldsymbol{c}(\boldsymbol{x})\boldsymbol{q} + \mathbf{grad}\, u = 0, \quad \text{on } \Omega, \tag{4.1}$$

$$\operatorname{div}\, \boldsymbol{q} = f \quad \text{on } \Omega, \tag{4.2}$$

$$u = g \quad \text{on } \partial\Omega, \tag{4.3}$$

where $\boldsymbol{c}(\boldsymbol{x})$ is a uniformly positive definite $N \times N$ matrix function on Ω with components in $L^\infty(\Omega)$. Now consider the following mixed method that provides numerical approximations $\boldsymbol{q}_p$ and u_p for the exact solution components $\boldsymbol{q}$ and u respectively. The spectral Raviart-Thomas mixed method defines $(\boldsymbol{q}_p, u_p) \in \boldsymbol{R}_p \times P_p$ by

$$(\boldsymbol{c}\,\boldsymbol{q}_p, \boldsymbol{r}) - (u_p, \operatorname{div} \boldsymbol{r}) = -\int_{\partial\Omega} g\, \boldsymbol{r} \cdot \boldsymbol{n}\, \mathrm{d}s, \quad \text{for all } \boldsymbol{r} \in \boldsymbol{R}_p \tag{4.4}$$

$$(v, \operatorname{div} \boldsymbol{q}_p) = \int_\Omega f v \, \mathrm{d}x, \quad \text{for all } v \in P_p. \tag{4.5}$$

To prove the quasioptimality of the method, we need to verify (1.2) and (1.3). Condition (1.3) obviously holds for this problem. Verification of (1.2) is done in the next theorem.

Theorem 4.1. *There exists a positive constant $\mathcal{C}_2$ independent of p such that*

$$\|v\|_{0,\Omega} \le \mathcal{C}_2 \sup_{\boldsymbol{r}_p \in \boldsymbol{R}_p} \frac{(v, \operatorname{div} \boldsymbol{r}_p)_\Omega}{\|\boldsymbol{r}_p\|_{\boldsymbol{H}(\operatorname{div},\Omega)}}, \quad \textit{for all } v \in P_p. \tag{4.6}$$

Proof. To prove the inf-sup condition (4.6), let $v \in P_p$ and consider $w_v \in H_0^1(\Omega)$ that solves

$$-\Delta w_v = v.$$

Then, by Poincaré inequality,

$$\|\,\mathbf{grad}\, w_v\|_{0,\Omega}^2 = (v, w_v) \le \mathcal{C}_{\mathrm{Pr}} \|v\|_{0,\Omega} \|\,\mathbf{grad}\, w_v\|_{0,\Omega}.$$

Therefore, $\boldsymbol{r} \equiv -\operatorname{\mathbf{grad}} w_v$ satisfies

$$\begin{aligned}\|\boldsymbol{r}\|^2_{\boldsymbol{H}(\mathrm{div},\Omega)} &= \|\operatorname{\mathbf{grad}} w_v\|^2_{0,\Omega} + \|\Delta w_v\|^2_{0,\Omega} \le (\mathcal{C}^2_{\mathrm{Pr}} + 1)\|v\|^2_{0,\Omega},\\ \operatorname{div} \boldsymbol{r} &= v.\end{aligned}$$

As a consequence of the commuting diagram of Theorem 3.1, $\operatorname{div} \boldsymbol{\Pi}^R_p \boldsymbol{r} = \Pi_p \operatorname{div} \boldsymbol{r} = v$, so

$$\begin{aligned}\sup_{\boldsymbol{r}\in\boldsymbol{R}_p} \frac{(v, \operatorname{div} \boldsymbol{r}_p)}{\|\boldsymbol{r}_p\|_{\boldsymbol{H}(\mathrm{div},\Omega)}} &\ge \frac{(v, \operatorname{div} \boldsymbol{\Pi}^R_p \boldsymbol{r})}{\|\boldsymbol{\Pi}^R_p \boldsymbol{r}\|_{\boldsymbol{H}(\mathrm{div},\Omega)}} \ge \frac{\|v\|^2_{0,\Omega}}{(1+\mathcal{C}^2_D)^{1/2}\|\boldsymbol{r}\|_{\boldsymbol{H}(\mathrm{div},\Omega)}}\\ &\ge \frac{\|v\|^2_{0,\Omega}}{(1+\mathcal{C}^2_D)^{1/2}(1+\mathcal{C}^2_{\mathrm{Pr}})^{1/2}\|v\|_{0,\Omega}} = \frac{1}{\mathcal{C}_2}\|v\|_{0,\Omega}.\end{aligned}$$

Thus the inf-sup condition follows. □

The general technique of using the exact solution of a boundary value problem to prove an inf-sup condition employed in the proof above is standard (cf. [3, Proposition 2.8, Chapter II]). The new ingredient above is the use of a p optimal projector. The quasioptimality estimate for the method now follows (cf. (1.1)).

5. Discrete Friedrichs type inequalities

In this section we prove inequalities of the type

$$\|\boldsymbol{q}\|_{0,\Omega} \le \mathcal{C}\|\operatorname{\mathbf{curl}} \boldsymbol{q}\|_{0,\Omega}$$

for $\boldsymbol{q}$ in appropriate spaces. Obviously such inequalities cannot hold in spaces with gradient vector fields. It is also obvious from the exactness of sequence (2.1) that such an inequality holds for all functions in the orthogonal complement of $\boldsymbol{H}(\operatorname{\mathbf{curl}} 0, \Omega)$ in $\boldsymbol{H}(\operatorname{\mathbf{curl}}, \Omega)$. Similarly, by the exactness of (3.1), the inequality also holds in the orthogonal complement of $\boldsymbol{Q}^0_p$ in $\boldsymbol{Q}_p$, but in this case we cannot conclude merely from the exactness that the constant $\mathcal{C}$ is independent of p. We will prove that $\mathcal{C}$ is indeed independent of p. Inequalities proved here are useful in a full hp analysis of Maxwell discretizations [8] and in the next section.

Theorem 5.1. *Let $\boldsymbol{Q}^\perp_p = \{\boldsymbol{q} \in \boldsymbol{Q}_p : (\boldsymbol{q}, \operatorname{\mathbf{grad}} w) = 0$ for all $w \in P_{p+1}\}$. Then,*

$$\|\boldsymbol{q}\|_{0,\Omega} \le \mathcal{C}_K\|\operatorname{\mathbf{curl}} \boldsymbol{q}\|_{0,\Omega} \quad \text{for all } \boldsymbol{q} \in \boldsymbol{Q}^\perp_p,$$

where $\mathcal{C}_K$ is as in (2.9).

Proof. Since $\boldsymbol{q} \in \boldsymbol{Q}^\perp_p$,

$$\|\boldsymbol{q}\|_{0,\Omega} = \inf_{w\in P_{p+1}} \|\boldsymbol{q} - \operatorname{\mathbf{grad}} w\|_{0,\Omega}.$$

Furthermore, since $\operatorname{\mathbf{curl}}(\boldsymbol{q} - \boldsymbol{K}\operatorname{\mathbf{curl}} \boldsymbol{q}) = 0$, by the exactness of the sequence (3.1), there exists a $w \in P_{p+1}$ such that $\operatorname{\mathbf{grad}} w = \boldsymbol{q} - \boldsymbol{K}\operatorname{\mathbf{curl}} \boldsymbol{q}$. Therefore,

$$\begin{aligned}\|\boldsymbol{q}\|_{0,\Omega} &\le \|\boldsymbol{q} - (\boldsymbol{q} - \boldsymbol{K}\operatorname{\mathbf{curl}} \boldsymbol{q})\|_{0,\Omega}\\ &\le \mathcal{C}_K\|\operatorname{\mathbf{curl}} \boldsymbol{q}\|_{0,\Omega}.\end{aligned}$$

□

Remark 5.1. A similar inequality (with constant independent of polynomial degree) is proved in [12] for the hexahedral Nédélec space (in fact more generally for an *hp* Nédélec space based on a hexahedral mesh). The approach taken there is quite different: One proves a p approximation estimate and then uses it to obtain the discrete Friedrichs type inequality. In view of Remark 3.5, our approach gives an alternate proof of the inequality for the tensor product Nédélec space.

To prove a similar inequality involving spaces with boundary conditions, namely

$$\mathring{\boldsymbol{Q}}_p = \{\boldsymbol{q} \in \boldsymbol{Q}_p : \boldsymbol{n} \times \boldsymbol{q} = 0 \text{ on } \partial\Omega\},$$
$$\mathring{P}_{p+1} = \{q \in P_{p+1} : q = 0 \text{ on } \partial\Omega\},$$

we find that the above simple proof does not apply because $\boldsymbol{K}$ does not preserve homogeneous boundary conditions. We will need to first define an analogue of $\boldsymbol{K} : \boldsymbol{H}(\operatorname{div} 0, \Omega) \mapsto \boldsymbol{H}(\mathbf{curl}\,, \Omega)$, namely

$$\mathring{\boldsymbol{K}} : \mathring{\boldsymbol{H}}(\operatorname{div} 0, \Omega) \mapsto \mathring{\boldsymbol{H}}(\mathbf{curl}\,, \Omega),$$

where

$$\mathring{\boldsymbol{H}}(\operatorname{div} 0, \Omega) = \{\boldsymbol{v} \in \boldsymbol{H}(\operatorname{div} 0, \Omega) : \boldsymbol{v} \cdot \boldsymbol{n} = 0\},$$
$$\mathring{\boldsymbol{H}}(\mathbf{curl}\,, \Omega) = \{\boldsymbol{q} \in \boldsymbol{H}(\mathbf{curl}\,, \Omega) : \boldsymbol{n} \times \boldsymbol{q} = 0 \text{ on } \partial\Omega\}.$$

Here $\boldsymbol{n}$ denotes the outward unit normal on $\partial\Omega$.

We now need to assume that Ω is a tetrahedron because we use a polynomial extension result currently available on tetrahedra. To describe this result, let $T_p(\partial\Omega) = \{v : v$ is continuous on $\partial\Omega$ and v is a polynomial of degree at most p on each face of the tetrahedron $\Omega\}$. It is proved in [13] that there exists an extension operator $\mathcal{E} : H^{1/2}(\partial\Omega) \mapsto H^1(\Omega)$ such that

(1) the trace of $\mathcal{E}v$ on $\partial\Omega$ coincides with v,
(2) there is a constant $\mathcal{C}_{\text{ext}}$ independent of v such that

$$\|\mathcal{E}v\|_{H^1(\Omega)} \le \mathcal{C}_{\text{ext}} \|v\|_{H^{1/2}(\partial\Omega)} \quad \text{for all } v \in H^{1/2}(\partial\Omega), \tag{5.1}$$

(3) and whenever $v \in T_p(\partial\Omega)$ the extension $\mathcal{E}v \in P_p$.

In the proof of the next theorem where we construct the required $\mathring{\boldsymbol{K}}$, we denote the tangential component of any vector field $\boldsymbol{q}$ on $\partial\Omega$ by $\boldsymbol{q}_\mathsf{T}$, i.e., $\boldsymbol{q}_\mathsf{T} = \boldsymbol{q} - (\boldsymbol{q} \cdot \boldsymbol{n})\boldsymbol{n}$. The same subscript will distinguish tangential differential operators on $\partial\Omega$. For definitions of surface gradient, curl(s), divergence, etc., on nonsmooth surfaces see [4]. Let $\mathring{\boldsymbol{R}}_p = \{\boldsymbol{r} \in \boldsymbol{R}_p : \boldsymbol{r} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega\}$.

Theorem 5.2. *Let Ω be a tetrahedron. Then there exists an operator $\mathring{\boldsymbol{K}}$ on $\mathring{\boldsymbol{H}}(\operatorname{div} 0, \Omega)$ with the following properties:*

(1) $\mathbf{curl}\, \mathring{\boldsymbol{K}}\boldsymbol{v} = \boldsymbol{v}$ *for all* $\boldsymbol{v} \in \mathring{\boldsymbol{H}}(\operatorname{div} 0, \Omega)$.
(2) $\boldsymbol{n} \times \mathring{\boldsymbol{K}}\boldsymbol{v} = 0$ *on* $\partial\Omega$ *for all* $\boldsymbol{v} \in \mathring{\boldsymbol{H}}(\operatorname{div} 0, \Omega)$.
(3) *There exists a constant* $\mathring{\mathcal{C}}_K > 0$ *independent of* $\boldsymbol{v}$ *such that*

$$\|\mathring{\boldsymbol{K}}\boldsymbol{v}\|_{0,\Omega} \le \mathring{\mathcal{C}}_K \|\boldsymbol{v}\|_{0,\Omega} \quad \text{for all } \boldsymbol{v} \in \mathring{\boldsymbol{H}}(\operatorname{div} 0, \Omega). \tag{5.2}$$

(4) *Whenever* $\boldsymbol{v}$ *is in* $\mathring{\boldsymbol{R}}_p$, *the function* $\mathring{\boldsymbol{K}}\boldsymbol{v}$ *is in* $\mathring{\boldsymbol{Q}}_p$.

Proof. We will construct $\mathring{\boldsymbol{K}}\boldsymbol{v}$ by subtracting an appropriate gradient field from $\boldsymbol{K}\boldsymbol{v}$. In order to find out what gradient field is appropriate, note first that since $\boldsymbol{K}$ is a right inverse of curl and $\boldsymbol{v} \in \mathring{\boldsymbol{H}}(\operatorname{div} 0, \Omega)$,

$$\boldsymbol{n} \cdot (\mathbf{curl}\, \boldsymbol{K}\boldsymbol{v}) = \boldsymbol{n} \cdot \boldsymbol{v} = 0 \text{ on } \partial\Omega.$$

Since $\boldsymbol{n} \cdot (\mathbf{curl}\, \boldsymbol{K}\boldsymbol{v}) = \operatorname{div}_{\mathsf{T}}(\boldsymbol{K}\boldsymbol{v} \times \boldsymbol{n})$, by the Hodge decomposition on $\partial\Omega$ established in [5], we find that $\boldsymbol{K}\boldsymbol{v} \times \boldsymbol{n} = \mathbf{curl}_{\mathsf{T}}\phi_{\boldsymbol{v}}$ for some $\phi_{\boldsymbol{v}} \in H^{1/2}(\partial\Omega)$. In other words, since $\mathbf{curl}_{\mathsf{T}}\phi_{\boldsymbol{v}} = \mathbf{grad}_{\mathsf{T}}\phi_{\boldsymbol{v}} \times \boldsymbol{n}$,

$$\boldsymbol{n} \times (\boldsymbol{K}\boldsymbol{v} \times \boldsymbol{n}) = \boldsymbol{n} \times (\mathbf{grad}_{\mathsf{T}}\phi_{\boldsymbol{v}} \times \boldsymbol{n}) = \mathbf{grad}_{\mathsf{T}}\phi_{\boldsymbol{v}},$$

or equivalently, the tangential component of $\boldsymbol{K}\boldsymbol{v}$ satisfies

$$(\boldsymbol{K}\boldsymbol{v})_{\mathsf{T}} = \mathbf{grad}_{\mathsf{T}}\phi_{\boldsymbol{v}} \quad \text{on } \partial\Omega. \tag{5.3}$$

Define

$$\mathring{\boldsymbol{K}}\boldsymbol{v} = \boldsymbol{K}\boldsymbol{v} - \mathbf{grad}\, \mathcal{E}\phi_{\boldsymbol{v}}.$$

By construction, we immediately see that Statements (1) and (2) of the theorem hold.

To prove Statement (3), we use (5.1). Since the extension operator $\mathcal{E}$ preserves constants, it follows that for any constant $c \in \mathbb{R}$, we have $\|\mathbf{grad}\, \mathcal{E}\phi_{\boldsymbol{v}}\|_{0,\Omega} = \|\mathbf{grad}\, \mathcal{E}(\phi_{\boldsymbol{v}} - c)\|_{0,\Omega} \le \|\phi_{\boldsymbol{v}} - c\|_{H^{1/2}(\partial\Omega)}$, so

$$\|\mathbf{grad}\, \mathcal{E}\phi_{\boldsymbol{v}}\|_{0,\Omega} \le \mathcal{C}_{\text{ext}}\|\phi_{\boldsymbol{v}}\|_{H^{1/2}(\partial\Omega)/\mathbb{R}}.$$

Hence

$$\begin{aligned} \|\mathring{\boldsymbol{K}}\boldsymbol{v}\|_{0,\Omega} &\le \|\boldsymbol{K}\boldsymbol{v}\|_{0,\Omega} + \|\mathbf{grad}\, \mathcal{E}\phi_{\boldsymbol{v}}\|_{0,\Omega} \\ &\le \mathcal{C}_K\|\boldsymbol{v}\|_{0,\Omega} + \mathcal{C}_{\text{ext}}\|\phi_{\boldsymbol{v}}\|_{H^{1/2}(\partial\Omega)/\mathbb{R}}. \end{aligned} \tag{5.4}$$

Now, by the exact sequence property of boundary spaces established in [5], we find that

$$\mathbf{grad}_{\mathsf{T}} : H^{1/2}(\partial\Omega)/\mathbb{R} \mapsto \boldsymbol{H}_{\perp}^{-1/2}(\partial\Omega)$$

is an injective operator whose range is closed in $\boldsymbol{H}_{\perp}^{-1/2}(\partial\Omega)$. (The space $\boldsymbol{H}_{\perp}^{-1/2}(\partial\Omega)$ and its norm is as defined in [5].) Hence there exists a constant $\mathcal{C}_{\text{grad}}$ such that

$$\|\phi\|_{H^{1/2}(\partial\Omega)/\mathbb{R}} \le \mathcal{C}_{\text{grad}}\|\mathbf{grad}_{\mathsf{T}}\phi\|_{\boldsymbol{H}_{\perp}^{-1/2}(\partial\Omega)} \quad \text{for all } \phi \in H^{1/2}(\partial\Omega).$$

Using this in (5.4), we now have that

$$\|\mathring{\boldsymbol{K}}\boldsymbol{v}\|_{0,\Omega} \le \mathcal{C}_K\|\boldsymbol{v}\|_{0,\Omega} + \mathcal{C}_{\text{ext}}\mathcal{C}_{\text{grad}}\|\mathbf{grad}_{\mathsf{T}}\phi_{\boldsymbol{v}}\|_{\boldsymbol{H}_{\perp}^{-1/2}(\partial\Omega)}.$$

which, by virtue of (5.3), implies that

$$\|\mathring{\boldsymbol{K}}\boldsymbol{v}\|_{0,\Omega} \le \mathcal{C}_K\|\boldsymbol{v}\|_{0,\Omega} + \mathcal{C}_{\text{ext}}\mathcal{C}_{\text{grad}}\|(\boldsymbol{K}\boldsymbol{v})_{\mathsf{T}}\|_{\boldsymbol{H}_{\perp}^{-1/2}(\partial\Omega)}. \tag{5.5}$$

To complete the proof of Statement (3), we need a final ingredient: It is proved in [4, Theorem 3.10] that there exists a constant $\mathcal{C}_{\text{trace}}$ independent of $\boldsymbol{q}$ such that

$$\|\boldsymbol{q}_{\mathsf{T}}\|_{\boldsymbol{H}_{\perp}^{-1/2}(\partial\Omega)} \le \mathcal{C}_{\text{trace}}\|\boldsymbol{q}\|_{\boldsymbol{H}(\mathbf{curl}\,,\Omega)}, \quad \text{for all } \boldsymbol{q} \in \boldsymbol{H}(\mathbf{curl}\,, \Omega).$$

This result together with (5.5) yields

$$\begin{aligned} \|\mathring{\boldsymbol{K}}\boldsymbol{v}\|_{0,\Omega} &\le \mathcal{C}_K\|\boldsymbol{v}\|_{0,\Omega} + \mathcal{C}_{\text{ext}}\mathcal{C}_{\text{grad}}\mathcal{C}_{\text{trace}}\|\boldsymbol{K}\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl}\,,\Omega)} \\ &\le (\mathcal{C}_K + (\mathcal{C}_K^2 + 1)^{1/2}\mathcal{C}_{\text{ext}}\mathcal{C}_{\text{grad}}\mathcal{C}_{\text{trace}})\|\boldsymbol{v}\|_{0,\Omega}, \end{aligned}$$

where we have also used the fact that $\boldsymbol{K}$ is a right inverse of curl. Thus the proof of (5.2) is complete.

It now only remains to prove Statement (4) of the theorem. It suffices to prove that $\phi_{\boldsymbol{v}} \in T_{p+1}(\partial\Omega)$ because, it then follows, by the nature of the extension operator, that $\mathbf{grad}\,\mathcal{E}\phi_{\boldsymbol{v}} \in \boldsymbol{P}_p$. Let e be a face of the tetrahedron Ω. We will now show that $\phi_{\boldsymbol{v}}$ is a polynomial of degree at most $p+1$ on e. By (5.3), $\phi_{\boldsymbol{v}}$ is a polynomial of degree at most $p+2$ on e. To compare the highest order terms in (5.3), let $\phi_{\boldsymbol{v}}^{(p+2)}$ denote the sum of terms in $\phi_{\boldsymbol{v}}$ of degree equal to $p+2$ in the components of $\boldsymbol{x}_{\mathsf{T}}$ and let $(\boldsymbol{K}\boldsymbol{v})_{\mathsf{T}}^{(p+1)}$ denote the sum of terms in $(\boldsymbol{K}\boldsymbol{v})_{\mathsf{T}}$ of degree $p+1$. Since $\boldsymbol{v}$ is in $\mathring{\boldsymbol{R}}_p$, the integral

$$\boldsymbol{k}(\boldsymbol{x}) = \int_0^1 t^3 \boldsymbol{v}(t^2(\boldsymbol{x}-\boldsymbol{a})+\boldsymbol{a})\,\mathrm{d}t$$

defines a function that can be decomposed as $\boldsymbol{k} = (\boldsymbol{x}-\boldsymbol{a})\widetilde{q}_p + \boldsymbol{q}_p$ for some homogeneous polynomials $\widetilde{q}_p$ and some $\boldsymbol{q}_p \in \boldsymbol{P}_p$. Hence, denoting $\boldsymbol{q}_p^{(\boldsymbol{n})} = \boldsymbol{n}(\boldsymbol{q}_p\cdot\boldsymbol{n})$, we have

$$\begin{aligned}
-\frac{1}{2}\boldsymbol{K}\boldsymbol{v} &= (\boldsymbol{x}-\boldsymbol{a})\times((\boldsymbol{x}-\boldsymbol{a})\widetilde{q}_p + \boldsymbol{q}_p)\\
&= \boldsymbol{x}_{\mathsf{T}}\times\boldsymbol{q}_p^{(\boldsymbol{n})} - a_{\mathsf{T}}\times\boldsymbol{q}_p^{(\boldsymbol{n})} + (\boldsymbol{x}-\boldsymbol{a})_{\mathsf{T}}\times(\boldsymbol{q}_p)_{\mathsf{T}} + (\boldsymbol{n}\times\boldsymbol{q}_p)(\boldsymbol{x}-\boldsymbol{a})\cdot\boldsymbol{n}
\end{aligned}$$

Since $(\boldsymbol{x}-\boldsymbol{a})\cdot\boldsymbol{n}$ is constant on e and $(\boldsymbol{x}-\boldsymbol{a})_{\mathsf{T}}\times(\boldsymbol{q}_p)_{\mathsf{T}}$ is in the $\boldsymbol{n}$-direction, we have

$$(\boldsymbol{K}\boldsymbol{v})_{\mathsf{T}}^{(p+1)} = -2\boldsymbol{x}_{\mathsf{T}}\times\boldsymbol{q}_p^{(\boldsymbol{n})}, \quad \text{on } e.$$

By (5.3),

$$\mathbf{grad}_{\mathsf{T}}\phi_{\boldsymbol{v}}^{(p+2)} = -2\boldsymbol{x}_{\mathsf{T}}\times\boldsymbol{q}_p^{(\boldsymbol{n})}, \quad \text{on } e.$$

Taking the innerproduct of both sides of the above equation with $\boldsymbol{x}_{\mathsf{T}}$ and using Euler's identity

$$(p+2)\,\boldsymbol{x}_{\mathsf{T}}\cdot\mathbf{grad}_{\mathsf{T}}\phi_{\boldsymbol{v}}^{(p+2)} = \phi_{\boldsymbol{v}}^{(p+2)},$$

we find that $\phi_{\boldsymbol{v}}^{(p+2)} = 0$. Consequently, $\phi_{\boldsymbol{v}}$ is a polynomial of degree at most $p+1$ on e.

The above argument applies to every face of Ω, so we have proved that $\phi_{\boldsymbol{v}}$ restricted to each face of Ω is a polynomial of degree at most $p+1$. Since we also know that $\phi_{\boldsymbol{v}} \in H^{1/2}(\partial\Omega)$, it is easy to see from the integrals defining the $H^{1/2}(\partial\Omega)$-seminorm that $\phi_{\boldsymbol{v}}$ is continuous on $\partial\Omega$. Thus $\phi_{\boldsymbol{v}} \in T_{p+1}(\partial\Omega)$. ☐

Theorem 5.3. *Let* $\mathring{\boldsymbol{Q}}_p^{\perp} = \{\boldsymbol{q}\in\mathring{\boldsymbol{Q}}_p : (\boldsymbol{q},\mathbf{grad}\,v) = 0$ *for all* $v \in \mathring{P}_{p+1}\}$. *Then,*

$$\|\boldsymbol{q}\|_{0,\Omega} \le \mathring{\mathcal{C}}_K\|\mathbf{curl}\,\boldsymbol{q}\|_{0,\Omega}, \quad \text{for all } \boldsymbol{q}\in\mathring{\boldsymbol{Q}}_p^{\perp},$$

where $\mathring{\mathcal{C}}_K$ *is the constant in (5.2).*

Proof. The proof proceeds just like the proof of Theorem 5.1, but using $\mathring{\boldsymbol{K}}$ in place of $\boldsymbol{K}$. ☐

6. Application to Maxwell equations

We now apply the results of the previous section to two mixed variational problems arising from systems of the type

$$\begin{aligned} \mathbf{curl}\,\mu^{-1}\mathbf{curl}\,\boldsymbol{E} &= \boldsymbol{J} \quad \text{on } \Omega, \\ \operatorname{div} \boldsymbol{E} &= 0 \quad \text{on } \Omega. \end{aligned} \tag{6.1}$$

Such equations arise when computing vector potentials in magnetostatics. Here μ is a positive bounded function in $L^\infty(\Omega)$ satisfying $\mu \geq \mu_0$ for some $\mu_0 > 0$ and $\boldsymbol{J} \in \boldsymbol{H}(\operatorname{div} 0, \Omega)$. For the purposes of analysis we assume that Ω is a tetrahedron throughout this section. Among the usually occurring boundary conditions, the simplest is

$$\boldsymbol{n} \times \boldsymbol{E} = 0 \quad \text{on } \partial\Omega. \tag{6.2}$$

A well known mixed variational formulation [11] of (6.1) and (6.2) is obtained by introducing a Lagrange multiplier $\psi \in H_0^1(\Omega)$: Find $\boldsymbol{E} \in \mathring{\boldsymbol{H}}(\mathbf{curl}\,, \Omega)$ and $\psi \in H_0^1(\Omega)$ satisfying

$$(\mu^{-1}\mathbf{curl}\,\boldsymbol{E}, \mathbf{curl}\,\boldsymbol{q}) - (\mathbf{grad}\,\psi, \boldsymbol{q}) = (\boldsymbol{J}, \boldsymbol{q}), \tag{6.3}$$

$$(\mathbf{grad}\,w, \boldsymbol{E}) = 0, \tag{6.4}$$

for all $\boldsymbol{q} \in \mathring{\boldsymbol{H}}(\mathbf{curl}\,, \Omega)$ and $w \in H_0^1(\Omega)$. Note that the second term in (6.3) makes the formulation symmetric. Clearly, this term is zero because $\operatorname{div} \boldsymbol{J} = 0$.

Another mixed formulation for (6.1) can be obtained by incorporating the divergence free condition of (6.1) into the polynomial spaces [2] . It is motivated by the following first order reformulation of (6.1):

$$\boldsymbol{H} = \mu^{-1}\mathbf{curl}\,\boldsymbol{E}, \quad \mathbf{curl}\,\boldsymbol{H} \quad = \boldsymbol{J}.$$

If we impose the magnetic symmetry wall boundary condition

$$\boldsymbol{n} \times \mu^{-1}\mathbf{curl}\,\boldsymbol{E} = 0 \quad \text{on } \partial\Omega, \tag{6.5}$$

the mixed formulation is to find $\boldsymbol{H} \in \mathring{\boldsymbol{H}}(\mathbf{curl}\,, \Omega)$ and $\boldsymbol{E} \in \mathring{\boldsymbol{H}}(\operatorname{div} 0, \Omega)$ satisfying

$$(\mu\boldsymbol{H}, \boldsymbol{q}) - (\boldsymbol{E}, \mathbf{curl}\,\boldsymbol{q}) = 0, \tag{6.6}$$

$$(\boldsymbol{r}, \mathbf{curl}\,\boldsymbol{H}) = (\boldsymbol{J}, \boldsymbol{r}), \tag{6.7}$$

for all $\boldsymbol{q} \in \mathring{\boldsymbol{H}}(\mathbf{curl}\,, \Omega)$ and all $\boldsymbol{r} \in \mathring{\boldsymbol{H}}(\operatorname{div} 0, \Omega)$. In the case of the boundary condition (6.2), the same equations hold on analogous spaces without boundary conditions. (An analysis simpler than the ensuing one holds for analogous formulations on spaces without boundary conditions, but we shall not discuss it.) The following are spectral discretizations of these variational formulations:

Problem 6.1. Find $(\boldsymbol{E}_p, \psi_p) \in \mathring{\boldsymbol{Q}}_p \times \mathring{P}_{p+1}$ $(p \geq 3)$ such that

$$\begin{aligned} (\mu^{-1}\mathbf{curl}\,\boldsymbol{E}_p, \mathbf{curl}\,\boldsymbol{q}_p) - (\mathbf{grad}\,\psi_p, \boldsymbol{q}_p) &= (\boldsymbol{J}, \boldsymbol{q}_p), && \text{for all } \boldsymbol{p} \in \mathring{\boldsymbol{Q}}_p, \\ (\mathbf{grad}\,w_p, \boldsymbol{E}_p) &= 0, && \text{for all } w_p \in \mathring{P}_{p+1}. \end{aligned}$$

Problem 6.2. Find $(\boldsymbol{H}_p, \boldsymbol{E}_p) \in \mathring{\boldsymbol{Q}}_p \times \mathring{\boldsymbol{R}}_p^0$ such that

$$\begin{aligned} (\mu\boldsymbol{H}_p, \boldsymbol{q}) - (\boldsymbol{E}_p, \mathbf{curl}\,\boldsymbol{q}) &= 0, && \text{for all } \boldsymbol{q} \in \mathring{\boldsymbol{Q}}_p, \\ (\boldsymbol{r}, \mathbf{curl}\,\boldsymbol{H}_p) &= (\boldsymbol{J}, \boldsymbol{r}), && \text{for all } \boldsymbol{r} \in \mathring{\boldsymbol{R}}_p^0, \end{aligned}$$

where $\mathring{\boldsymbol{R}}_p^0 = \{\boldsymbol{r} \in \boldsymbol{R}_p^0 : \boldsymbol{r}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\}$.

While Problem 6.1 discretizes (6.3)–(6.4), Problem 6.2 discretizes (6.6)–(6.7). We will now prove the quasioptimality of the two methods. To analyze Problem 6.1, as before, we need to verify two conditions. The first condition, namely

$$\|w_p\|_{H^1(\Omega)} \le \mathcal{C} \sup_{\boldsymbol{q}_p\in\mathring{\boldsymbol{Q}}_p} \frac{(\mathbf{grad}\, w_p, \boldsymbol{q}_p)}{\|\boldsymbol{q}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}} \quad \text{for all } w_p \in \mathring{P}_{p+1},$$

follows from the imbedding

$$\mathbf{grad}\, \mathring{P}_{p+1} \subset \mathring{\boldsymbol{Q}}_p,$$

and the Poincaré inequality. Condition (1.3) follows from Theorem 5.3. Thus we obtain a quasioptimality estimate. Moreover, since ψ_p approximates the zero Lagrange multiplier ψ, we have the following corollary.

Corollary 6.1. *If $\boldsymbol{E}$ satisfies (6.3)–(6.4) and $\mathring{\boldsymbol{E}}_p$ solves Problem 6.1, there is a constant $\mathcal{C}$ independent of p such that*

$$\|\boldsymbol{E} - \mathring{\boldsymbol{E}}_p\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \le \mathcal{C} \inf_{\boldsymbol{q}_p\in\mathring{\boldsymbol{Q}}_p} \|\boldsymbol{E} - \boldsymbol{q}_p\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}.$$

Finally, we discuss Problem 6.2. Condition (1.3) is trivial in this case. The required inf-sup condition can be proved using a projector similar to $\boldsymbol{\Pi}_p^Q$. Let $\mathring{\boldsymbol{\Pi}}_p^{Q0}$ and $\mathring{\boldsymbol{\Pi}}_p^{R0}$ denote the L^2-orthogonal projectors into $\mathring{\boldsymbol{Q}}_p^0 = \{\boldsymbol{q} \in \mathring{\boldsymbol{Q}}_p : \mathbf{curl}\,\boldsymbol{q} = 0\}$ and $\mathring{\boldsymbol{R}}_p^0$, respectively. Define

$$\mathring{\boldsymbol{\Pi}}_p^Q \boldsymbol{q} = \mathring{\boldsymbol{\Pi}}_p^{Q0}\boldsymbol{q} + (\boldsymbol{I} - \mathring{\boldsymbol{\Pi}}_p^{Q0})\mathring{\boldsymbol{K}}(\mathring{\boldsymbol{\Pi}}_p^{R0}\mathbf{curl}\,\boldsymbol{q}). \tag{6.8}$$

Then we have the following inf-sup condition.

Theorem 6.1. *There exists a positive constant $\mathcal{C}$ independent of p such that*

$$\|\boldsymbol{z}_p\|_{0,\Omega} \le \mathcal{C} \sup_{\boldsymbol{q}_p\in\mathring{\boldsymbol{Q}}_p} \frac{(\boldsymbol{z}_p, \mathbf{curl}\,\boldsymbol{q}_p)}{\|\boldsymbol{q}_p\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}}, \quad \textit{for all } \boldsymbol{z}_p \in \mathring{\boldsymbol{R}}_p^0, \tag{6.9}$$

Proof. For any given $\boldsymbol{z}_p \in \boldsymbol{R}_p^0$, let $\boldsymbol{q}$ be unique the solution of the following div-curl problem:

$$\begin{aligned} \mathbf{curl}\,\boldsymbol{q} &= \boldsymbol{z}_p, && \text{on } \Omega\\ \operatorname{div}\boldsymbol{q} &= 0, && \text{on } \Omega\\ \boldsymbol{q}\times\boldsymbol{n} &= 0, && \text{on } \partial\Omega. \end{aligned}$$

Then, by [9, Lemma 3.4], there exists a constant $\mathcal{C}_{\text{curl}}$ (independent of $\boldsymbol{z}_p$) such that

$$\|\boldsymbol{q}\|_{0,\Omega} \le \mathcal{C}_{\text{curl}}\|\mathbf{curl}\,\boldsymbol{q}\|_{0,\Omega}. \tag{6.10}$$

Moreover, by Theorem 3.1,

$$\mathbf{curl}\,(\mathring{\boldsymbol{\Pi}}_p^Q\boldsymbol{q}) = \boldsymbol{\Pi}_p^R\mathbf{curl}\,\boldsymbol{q} = \boldsymbol{z}_p.$$

Hence,

$$\sup_{\boldsymbol{q}\in\mathring{\boldsymbol{Q}}_p} \frac{(\boldsymbol{z}_p, \mathbf{curl}\, \boldsymbol{q}_p)}{\|\boldsymbol{q}_p\|_{\boldsymbol{H}(\mathbf{curl}\,,\Omega)}} \geq \frac{(\boldsymbol{z}_p, \mathbf{curl}\,(\mathring{\boldsymbol{\Pi}}_p^Q \boldsymbol{q}))}{\|\mathring{\boldsymbol{\Pi}}_p^Q \boldsymbol{q}\|_{\boldsymbol{H}(\mathbf{curl}\,,\Omega)}} \geq \frac{\|\boldsymbol{z}_p\|_{0,\Omega}^2}{(1+\mathring{\mathcal{C}}_K^2)^{1/2}\|\boldsymbol{q}\|_{\boldsymbol{H}(\mathbf{curl}\,,\Omega)}},$$
$$\geq \frac{\|\boldsymbol{z}_p\|_{0,\Omega}^2}{(1+\mathring{\mathcal{C}}_K^2)^{1/2}(1+\mathcal{C}_{\mathrm{curl}}^2)^{1/2}\|\mathbf{curl}\,\boldsymbol{q}\|_{0,\Omega}} = \frac{\|\boldsymbol{z}_p\|_{0,\Omega}}{(1+\mathring{\mathcal{C}}_K^2)^{1/2}(1+\mathcal{C}_{\mathrm{curl}}^2)^{1/2}}.$$

Thus, the inf-sup condition (6.9) follows. □

Quasioptimality of Problem 6.2 now follows from Theorem 6.1.

7. Concluding remarks

We have constructed projectors that satisfy the commutativity properties important for mixed methods with norm bounds independent of p and discussed their applications to spectral mixed methods. The critical ingredients were the right inverse maps $\boldsymbol{D}, \boldsymbol{K}$, and G.

During the analysis of the two mixed problems with homogeneous tangential boundary conditions in Section 6, we also used another right inverse of curl, namely $\mathring{\boldsymbol{K}}$, which has the additional property that $\boldsymbol{n}\times\mathring{\boldsymbol{K}}\boldsymbol{v} = 0$ on $\partial\Omega$ whenever $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$. As we saw, this map can also be used to define a projector (namely $\mathring{\boldsymbol{\Pi}}_p^Q$; see (6.8)) on $\mathring{\boldsymbol{H}}(\mathbf{curl}\,,\Omega)$ that preserves zero tangential traces.

It is natural to ask if we can construct projectors analogous to $\mathring{\boldsymbol{\Pi}}_p^Q$ for the remaining spaces, say $\mathring{\boldsymbol{\Pi}}_p^R$ and $\mathring{\Pi}_p^W$, that preserves appropriate zero traces and satisfies the analogous commutativity and norm bound properties. The latter is easy: Fix $\boldsymbol{a}\in\partial\Omega$ and define

$$\mathring{\Pi}_p^W w = G\mathring{\boldsymbol{\Pi}}_p^{Q0}\,\mathbf{grad}\, w, \quad \text{for all } w\in H_0^1(\Omega).$$

Then $\mathring{\Pi}_p^W w \in \mathring{P}_{p+1}$, because the function $\boldsymbol{q} = \mathring{\boldsymbol{\Pi}}_p^{Q0}\,\mathbf{grad}\, w$ is irrotational, so the value of line integral that defines G, namely

$$\int_{\boldsymbol{a}}^{\boldsymbol{x}} \boldsymbol{q}\cdot \mathrm{d}t,$$

remains unchanged if we carry out the integration along any other path from $\boldsymbol{a}$ to $\boldsymbol{x}$. In particular, if $\boldsymbol{x}\in\partial\Omega$, we may choose to integrate along a curve lying entirely on the $\partial\Omega$ where $w = 0$, so $\mathring{\Pi}_p^W w = 0$ on $\partial\Omega$ whenever w vanishes on $\partial\Omega$.

However, the construction of $\mathring{\boldsymbol{\Pi}}_p^R$ appears to be more difficult. We would like to construct a bounded linear map $\mathring{\boldsymbol{D}} : L^2(\Omega)/\mathbb{R} \mapsto \mathring{\boldsymbol{H}}(\mathrm{div},\Omega)$ that is a right inverse of divergence along the lines of our construction of $\mathring{\boldsymbol{K}}$. But an analogue of the extension operator $\mathcal{E}$, namely a uniformly bounded polynomial extension operator from $\boldsymbol{H}_\perp^{-1/2}(\partial\Omega)$ into $\boldsymbol{H}(\mathbf{curl}\,,\Omega)$, is missing.

We established several inf-sup conditions independent of p. We also established two discrete Friedrichs type inequalities involving curl in Theorems 5.1 and 5.3. These inequalities are not only important for the analysis of the spectral mixed methods we considered, but are also the first step in a full hp-analysis of finite element discretizations for Maxwell equations. Although the right inverses introduced map polynomials into polynomials, they do not map piecewise polynomial spaces, such as p-finite elements spaces on fixed grids with more than one element, or *hp* finite element spaces, into similar spaces. Hence an *hp* analysis does not immediately follow from our projectors, but additional results needed are being explored [8].

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