INCOMPRESSIBLE FINITE ELEMENTS VIA HYBRIDIZATION. PART II: THE STOKES SYSTEM IN THREE SPACE DIMENSIONS

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ABSTRACT. We introduce a method that gives exactly incompressible velocity approximations to Stokes flow in three space dimensions. The method is designed by extending the ideas in Part I of this series, where the Stokes system in two space dimensions was considered. Thus we hybridize a vorticity-velocity formulation to obtain a new mixed method coupling approximations of tangential velocity and pressure on mesh faces. Once this relatively small tangential velocity–pressure system is solved, it is possible to recover a globally divergence-free numerical approximation of the fluid velocity, an approximation of the vorticity whose tangential component is continuous across inter-element boundaries, and a discontinuous numerical approximation of the pressure. The main difference with the two dimensional case treated in Part I is in the use of Nédélec elements, which necessitates development of new hybridization techniques. We also generalize the method to allow for varying polynomial degrees on different mesh elements and to incorporate certain non-standard but physically relevant boundary conditions.

1. INTRODUCTION

This is a sequel to our paper [7] in which we introduced a new hybridized method for the Stokes equations in two space dimensions. Here we generalize the ideas presented in [7] to the Stokes system in three space dimensions. We also extend the method to allow variable degrees of approximation on different mesh elements. As in [7], the three dimensional version of our method simultaneously yields an exactly divergence free numerical approximation of the fluid velocity and a continuous numerical approximation of the vorticity. A discontinuous numerical approximation of the pressure can also be recovered separately. These three approximations are obtained in an element by element fashion after one global system for certain Lagrange multipliers arising from the hybridization is solved. This global system represents a new "tangential velocity–pressure" discretization of the Stokes system on the mesh faces because the Lagrange multipliers are approximations to the pressure and tangential fluid velocity on element interfaces.

We are hybridizing a mixed formulation that has previously appeared in the literature [9, 14] (cf. [1, 3]). However, the previous works resort to introduction of a stream function variable to obtain exactly divergence free numerical velocities. This approach is beset with significant difficulties in three dimensions: (i) While the stream function is a scalar function in two dimensions, in three dimensions, it is a vector function, so its introduction into the method, as in [14], leads to a significant increase in number of degrees of freedom. (ii) The

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stream function is not uniquely defined. While in two dimensions it is defined up to a constant, in three dimensions, one has to impose a nontrivial "gauge condition". (iii) The definition of the stream function must take into account the topology of the three dimensional domain. For domains that are not simply connected, one must find "cuts" and base the definition of finite element spaces for the stream function on them – see [1]. Finding such cuts in automatic computation is not easy. (iv) Formulations involving the stream function alone leads to fourth order problems (see e.g. [1, 9]) and hence to badly conditioned matrices. Notwithstanding these difficulties, the use of the stream function has hitherto been the only successful approach in obtaining exactly incompressible approximations of all orders in three dimensions. The search for exactly incompressible numerical approximations to Stokes flow has a rich history. References to some previous attempts can be found in [4, 7, 11].

All the above mentioned difficulties disappear in our approach via hybridization. Because we do not introduce the stream function, our method requires nothing special to be done when the computational domain has nontrivial topology. For the same reason we never encounter a fourth order operator – our matrices represent discretizations of operators of second order only. Moreover, while the introduction of the stream function results in an increase in degrees of freedom in some of the previous works, our approach using hybridization actually results in a decrease in degrees of freedom, as we shall see in Section 3.

As we move from two to three space dimensions, the main difference we encounter is in the treatment of vorticity. When considering finite element approximations to vorticity, we now have to use the $H(\mathbf{curl}\,,\Omega)$-conforming Nédélec elements [13], while in two dimensions we used the simpler $H^1(\Omega)$-conforming finite elements. However, the velocity approximation is treated in exactly the same way as in the two dimensional case – it continues to be in an $H(\mathrm{div},\Omega)$-conforming subspace of exactly divergence free functions. Another important similarity between the two and three dimensional case is in the structure of the method and equations, so we are able to easily adapt the elimination procedure which we developed in [7] to three dimensions. The result is a Lagrange multiplier system that is completely analogous to the two dimensional case.

The introduction of Nédélec spaces necessitates development of new hybridization techniques in three dimensions. Indeed, the Nédélec space has edge degrees of freedom and none of the existing hybridization techniques handle them. To elaborate, consider the following sequence of spaces:

$$H^1(\Omega)/\mathbb{R} \xrightarrow{\ \mathbf{grad}\ } H(\mathbf{curl}\,,\Omega) \xrightarrow{\ \mathbf{curl}\ } H(\mathrm{div},\Omega) \xrightarrow{\ \mathrm{div}\ } L^2(\Omega).$$

As we traverse the sequence from right to left, the continuity conditions on the spaces become more complex. Finite element subspaces of $H(\mathrm{div},\Omega)$ consist of functions whose normal component is continuous across element interfaces. Hybridization techniques to relax such continuity are well known and they are the basis for the hybridized Raviart-Thomas and BDM-type methods [2, 5]. Such hybridizations relaxed continuity of finite element subspaces across interior mesh faces using traces from (just) two elements sharing an interior mesh face. However, once we move on to finite element subspaces of $H(\mathbf{curl}\,,\Omega)$, the continuity constraints are more complicated, as reflected by the fact that these spaces have edge degrees of freedom which are connected to multiple elements. Moving further left to $H^1(\Omega)$ we find finite element subspaces having vertex degrees of freedom, adding another layer of complexity. Since all previously known hybridization techniques relaxed continuity across mesh faces, we find a widespread belief that methods using edge and vertex degrees of freedom are not amenable to hybridization. In this paper, we dispel this belief by hybridizing a method that uses Nédélec spaces having edge degrees of freedom. It is also possible to hybridize methods that use $H^1(\Omega)$-subspaces, as we demonstrated in [7].

We make two other extensions in this paper. The first extends to the Stokes system what was done for second order elliptic equations in [6]. Thus, we exploit the ease of construction of variable degree methods via hybridization to give a variable degree version of the original mixed method. Our hybridized variable degree method does not require one to implement transition elements. This is quite convenient considering that transitional Nédélec elements are not trivial to implement. Second, we show how one can incorporate boundary conditions involving the pressure and tangential vorticity into our method. Although such boundary conditions are physically relevant, few methods are known that can incorporate them naturally.

We have kept the organization of this paper very similar to Part I [7] to render the analogies and differences with the two dimensional case transparent. We introduce the variable degree method in Section 2. In Section 3, we briefly present the elimination strategy to obtain a reduced Lagrange multiplier system. A computable basis for the space of Lagrange multipliers of variable degree is given in Section 4 and full details of the lowest order case are given in Section 5. Finally, in Section 6, we show how to incorporate other boundary conditions.

2. The variable degree hybridized mixed method

The three dimensional Stokes problem is to find a fluid velocity field $\boldsymbol{u}$ and pressure p satisfying

$$-\boldsymbol{\Delta u} + \mathbf{grad}\, p = \boldsymbol{f}, \quad \text{on } \Omega, \tag{1}$$

$$\operatorname{div} \boldsymbol{u} = 0, \quad \text{on } \Omega, \tag{2}$$

$$\boldsymbol{u} = \boldsymbol{g}, \quad \text{on } \partial\Omega. \tag{3}$$

Here we assume that Ω is a bounded connected domain with polyhedral boundary $\partial\Omega$ such that Ω lies only on one side of $\partial\Omega$ locally, the data $\boldsymbol{f}$ is in $L^2(\Omega)^3$ and $\boldsymbol{g} \in H^{1/2}(\partial\Omega)^3$. We do not assume that Ω is simply connected. We also do not assume that $\partial\Omega$ is connected. We require the data $\boldsymbol{g}$ to satisfy the compatibility condition

$$(g_n, 1)_{\partial\Omega} = 0,$$

where $g_n = \boldsymbol{g} \cdot \boldsymbol{n}$ and $\boldsymbol{n}$ is the outward unit normal on $\partial\Omega$. Under this assumption, it is well known that the Stokes problem has a unique solution.

Let us reformulate the Stokes problem by introducing vorticity $\boldsymbol{\omega} = \mathbf{curl}\, \boldsymbol{u}$. Using the identity

$$-\boldsymbol{\Delta u} = \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{u} - \mathbf{grad}\operatorname{div} \boldsymbol{u},$$

the Stokes system (1)–(3) can be rewritten as

$$\boldsymbol{\omega} - \mathbf{curl}\, \boldsymbol{u} = 0 \quad \text{on } \Omega, \tag{4}$$

$$\mathbf{curl}\, \boldsymbol{\omega} + \mathbf{grad}\, p = \boldsymbol{f} \quad \text{on } \Omega, \tag{5}$$

$$\operatorname{div} \boldsymbol{u} = 0, \quad \text{on } \Omega, \tag{6}$$

$$\boldsymbol{u}_{\mathsf{T}} = \boldsymbol{g}_{\mathsf{T}}, \quad \text{on } \partial\Omega, \tag{7}$$

$$\boldsymbol{u} \cdot \boldsymbol{n} = \boldsymbol{g} \cdot \boldsymbol{n}, \quad \text{on } \partial\Omega, \tag{8}$$

where we have split equation (3) into two equations, one in the direction of the outward unit normal $\boldsymbol{n}$ on $\partial\Omega$, and the other in the tangent plane, i.e., $\boldsymbol{g}_{\mathsf{T}} := \boldsymbol{g} - (\boldsymbol{g}\cdot\boldsymbol{n})\boldsymbol{n}$ denotes the tangential component of $\boldsymbol{g}$.

There is a well known weak problem based on this reformulation. Define $\mathcal{W} = H(\mathbf{curl}\,,\Omega)$ and

$$\mathcal{V}(b) = \{\boldsymbol{v} \in H(\mathrm{div},\Omega) : \mathrm{div}\,\boldsymbol{v} = 0 \text{ and } \boldsymbol{v}\cdot\boldsymbol{n}|_{\partial\Omega} = b\}.$$

for any $b \in H^{-1/2}(\partial\Omega)$. Then $(\boldsymbol{\omega},\boldsymbol{u})$ is the only element of $\mathcal{W}\times\mathcal{V}(g_n)$ satisfying

$$(\boldsymbol{\omega},\boldsymbol{\tau})_\Omega - (\boldsymbol{u},\mathbf{curl}\,\boldsymbol{\tau})_\Omega = (\boldsymbol{g}_{\mathsf{T}},\boldsymbol{\tau})_{\partial\Omega} \qquad \text{for all } \boldsymbol{\tau}\in\mathcal{W}, \tag{9}$$

$$(\boldsymbol{v},\mathbf{curl}\,\boldsymbol{\omega})_\Omega = (\boldsymbol{f},\boldsymbol{v})_\Omega \qquad \text{for all } \boldsymbol{v}\in\mathcal{V}(0). \tag{10}$$

Here $(\cdot,\cdot)_\Omega$ denotes the $L^2(\Omega)$ (or $L^2(\Omega)^3$) innerproduct. Note that the pressure has disappeared in this mixed formulation.

One way to develop a hybridized mixed method that discretizes (9)–(10) is to first approximate the weak formulation by a conforming mixed method and then relax the continuity constraints of the discrete spaces. Here, we motivate the construction of our variable degree hybridized mixed method (9)–(10) by another equivalent approach using the differential problem (4)–(8). Suppose the domain Ω is meshed by a tetrahedral mesh $\mathcal{T}$ (satisfying the usual finite element assumptions). To each tetrahedron K, we associate a degree $k(K)$ and the following pair of spaces:

$$W(K) = P_{k(K)}(K)^3 \oplus S_{k(K)+1}(K),$$
$$V(K) = \{\boldsymbol{v}\in P_{k(K)}(K)^3 :\ \mathrm{div}\,\boldsymbol{v} = 0\},$$

where $P_\ell(K)^3$ denotes the set of vector functions whose (three) components are polynomials of degree at most ℓ and $S_\ell(K)$ is the set of all vector functions $\boldsymbol{p}_\ell(\boldsymbol{x})$ whose components are homogeneous polynomials of degree ℓ satisfying $\boldsymbol{p}_\ell(\boldsymbol{x})\cdot\boldsymbol{x} = 0$. Define the variable degree Nédélec space with no continuity conditions by

$$W_h = \{\boldsymbol{w} : \boldsymbol{w}|_K \in W(K) \text{ for all } K\in\mathcal{T}\}.$$

While the vorticity is approximated in W_h, the velocity is approximated in

$$V_h = \{\boldsymbol{v} : \boldsymbol{v}|_K \in V(K) \text{ for all } K\in\mathcal{T}\}.$$

The numerical method is motivated by requiring that the equations (4) and (5) be satisfied weakly on each element K: Multiplying (4) and (5) by test functions $\boldsymbol{\tau}\in W(K)$ and $\boldsymbol{v}\in V(K)$, and integrating by parts,

$$(\boldsymbol{\omega},\boldsymbol{\tau})_K - (\boldsymbol{u},\mathbf{curl}\,\boldsymbol{\tau})_K - (\boldsymbol{u}_{\mathsf{T}},\boldsymbol{n}\times\boldsymbol{\tau})_{\partial K} = 0,$$
$$(\boldsymbol{v},\mathbf{curl}\,\boldsymbol{\omega})_K + (\boldsymbol{v}\cdot\boldsymbol{n},p)_{\partial K} = (\boldsymbol{f},\boldsymbol{v})_K,$$

where $\boldsymbol{u}_{\mathsf{T}}$ denotes the tangential component of $\boldsymbol{u}$ on ∂K. Therefore we require that the discrete approximations to vorticity and velocity, namely $\boldsymbol{\omega}_h$ and $\boldsymbol{u}_h$, respectively, satisfy

$$(\boldsymbol{\omega}_h,\boldsymbol{\tau})_K - (\boldsymbol{u}_h,\mathbf{curl}\,\boldsymbol{\tau})_K - (\boldsymbol{\lambda}_h,\boldsymbol{n}\times\boldsymbol{\tau})_{\partial K} = 0,$$
$$(\boldsymbol{v},\mathbf{curl}\,\boldsymbol{\omega}_h)_K + (\boldsymbol{v}\cdot\boldsymbol{n},p_h)_{\partial K} = (\boldsymbol{f},\boldsymbol{v})_K,$$

where we have introduced two additional approximations $\boldsymbol{\lambda}_h \approx \boldsymbol{u}_{\mathsf{T}}$ and $p_h \approx p$, which we shall call Lagrange multiplier approximations of the tangential velocity and pressure, respectively.

The description of the method is completed by adding appropriate continuity conditions for $\boldsymbol{\omega}_h$ and $\boldsymbol{u}_h$ at the element interfaces. Since $\boldsymbol{\omega}_h$ and $\boldsymbol{u}_h$ are to approximate $\boldsymbol{\omega}$ and $\boldsymbol{u}$

in (9)–(10), the functional setting of (9)–(10) clarifies the continuity constraints to be put on $\boldsymbol{\omega}_h$ and $\boldsymbol{u}_h$. To make this precise, let us introduce some more notation: Let $\mathcal{F}$ denote the set of all faces of the triangulation $\mathcal{T}$. On every interior face in $F \in \mathcal{F}$ shared by two tetrahedra K_F^+ and K_F^- we define

$$\llbracket \boldsymbol{v}\cdot\boldsymbol{n}\rrbracket_F = \boldsymbol{v}_F^+\cdot\boldsymbol{n}_F^+ + \boldsymbol{v}_F^-\cdot\boldsymbol{n}_F^-,$$
$$\llbracket \boldsymbol{n}\times\boldsymbol{v}\rrbracket_F = \boldsymbol{n}_F^+\times\boldsymbol{v}_F^+ + \boldsymbol{n}_F^-\times\boldsymbol{v}_F^-.$$

where $\boldsymbol{n}_F^+$ and $\boldsymbol{n}_F^-$ denote the outward unit normals on the boundaries of K_F^+ and K_F^-, respectively, and $\boldsymbol{v}_F^\pm(\boldsymbol{x}) = \lim_{\epsilon\downarrow 0}\boldsymbol{v}(\boldsymbol{x}-\epsilon\,\boldsymbol{n}_F^\pm)$. On faces $e\subset\partial\Omega$, we set

$$\llbracket \boldsymbol{v}\cdot\boldsymbol{n}\rrbracket_F = \boldsymbol{v}|_{\partial\Omega}\cdot\boldsymbol{n} \quad\text{ and }\quad \llbracket \boldsymbol{n}\times\boldsymbol{v}\rrbracket_F = 0.$$

By $\llbracket \boldsymbol{v}\cdot\boldsymbol{n}\rrbracket$ (without subscripts) we mean the function that is defined on the union of all the faces and equals $\llbracket \boldsymbol{v}\cdot\boldsymbol{n}\rrbracket_F$ on each face $e\in\mathcal{F}$. The function $\llbracket \boldsymbol{n}\times\boldsymbol{v}\rrbracket$ is similarly defined. Then here are our spaces of Lagrange multipliers:

$$P_h = \{p : \quad p = \llbracket \boldsymbol{v}\cdot\boldsymbol{n}\rrbracket \text{ for some } \boldsymbol{v}\in V_h\}, \tag{11}$$
$$M_h = \{\boldsymbol{\mu} : \quad \boldsymbol{\mu} = \llbracket \boldsymbol{n}\times\boldsymbol{v}\rrbracket \text{ for some } \boldsymbol{v}\in W_h\}. \tag{12}$$

They are ideal for imposing the natural continuity conditions of the Sobolev spaces $\mathcal{W}$ and $\mathcal{V}$ on the discrete approximations $\boldsymbol{\omega}_h$ and $\boldsymbol{v}_h$, e.g.,

$$\sum_{F\in\mathcal{F}}(\boldsymbol{\mu},\, \llbracket \boldsymbol{n}\times\boldsymbol{\omega}_h\rrbracket)_F = 0, \quad \text{for all } \boldsymbol{\mu}\in M_h,$$

implies that that $\boldsymbol{\omega}_h \in H(\mathbf{curl}\,)$.

Thus we have motivated the following definition of our variable degree hybridized mixed method: Find $(\boldsymbol{\omega}_h, \boldsymbol{u}_h, \boldsymbol{\lambda}_h, p_h) \in W_h\times V_h\times M_h\times P_h$ satisfying

$$(\boldsymbol{\omega}_h,\boldsymbol{\tau}_h)_\Omega - (\boldsymbol{u}_h, \mathbf{curl}\,\boldsymbol{\tau}_h)_\Omega - \sum_{F\in\mathcal{F}}(\boldsymbol{\lambda}_h,\, \llbracket \boldsymbol{n}\times\boldsymbol{\tau}_h\rrbracket)_F = (\boldsymbol{g}_{\mathsf{T}}, \boldsymbol{n}\times\boldsymbol{\tau}_h)_{\partial\Omega}, \tag{13}$$
$$(\boldsymbol{v}_h, \mathbf{curl}\,\boldsymbol{\omega}_h)_\Omega + \sum_{F\in\mathcal{F}}(p_h,\, \llbracket \boldsymbol{v}_h\cdot\boldsymbol{n}\rrbracket)_F = (\boldsymbol{f}, \boldsymbol{v}_h)_\Omega, \tag{14}$$
$$\sum_{F\in\mathcal{F}}(q_h,\, \llbracket \boldsymbol{u}_h\cdot\boldsymbol{n}\rrbracket)_F = (g_n, q_h)_{\partial\Omega}, \tag{15}$$
$$\sum_{F\in\mathcal{F}}(\boldsymbol{\mu}_h,\, \llbracket \boldsymbol{n}\times\boldsymbol{\omega}_h\rrbracket)_F = 0, \tag{16}$$

for all $\boldsymbol{\tau}_h\in W_h, \boldsymbol{v}_h\in V_h, q_h\in P_h, \boldsymbol{\mu}_h\in M_h$.

Proposition 2.1. *There is a unique solution for the system* (13)–(16).

Proof. Since the system is square, we only need to verify that when $\boldsymbol{f}$ and $\boldsymbol{g}$ are zero, all solution components vanish. Zero data implies that $\boldsymbol{\omega}_h$ and $\boldsymbol{u}_h$ lie in the following two spaces respectively:

$$\mathcal{W}_h = W_h\cap\mathcal{W}, \qquad \mathcal{V}_h(0) = V_h\cap\mathcal{V}(0).$$

Therefore we find from (13) and (14) that

$$(\boldsymbol{\omega}_h,\boldsymbol{\tau}_h)_\Omega - (\boldsymbol{u}_h, \mathbf{curl}\,\boldsymbol{\tau}_h)_\Omega = 0, \quad \text{for all } \boldsymbol{\tau}_h\in\mathcal{W}_h \tag{17}$$
$$(\boldsymbol{v}_h, \mathbf{curl}\,\boldsymbol{\omega}_h)_\Omega = 0, \quad \text{for all } \boldsymbol{v}_h\in\mathcal{V}_h(0). \tag{18}$$

Setting $\boldsymbol{v}_h = \boldsymbol{u}_h$ in (18) and adding these equations, one immediately finds that $(\boldsymbol{\omega}_h, \boldsymbol{\tau}_h)_\Omega = 0$ for all $\boldsymbol{\tau}_h \in \mathcal{W}_h$, so $\boldsymbol{\omega}_h = 0$. Now that $\boldsymbol{\omega}_h$ vanishes from (17), we have

$$(\boldsymbol{u}_h, \mathbf{curl}\, \boldsymbol{\tau}_h)_\Omega = 0 \quad \text{for all } \boldsymbol{\tau}_h \in \mathcal{W}_h. \tag{19}$$

By a well known property of the Nédélec space, we have that on each element $\mathbf{curl}\ W(K) = V(K)$. Moreover, $[\![\boldsymbol{n} \cdot \mathbf{curl}\, \boldsymbol{w}]\!]_F = 0$ whenever $[\![\boldsymbol{n} \times \boldsymbol{w}]\!]_F = 0$ for every interior mesh face F. Hence, it is easy to see that for the variable degree spaces $\mathcal{W}_h$ and $\mathcal{V}_h(0)$ we have (cf. [9, Lemma III.5.1])

$$\mathcal{V}_h(0) \subset \mathbf{curl}\ \mathcal{W}_h.$$

Therefore, in (19) we can choose $\boldsymbol{\tau}_h$ such that $\mathbf{curl}\, \boldsymbol{\tau}_h = \boldsymbol{u}_h$, so $\boldsymbol{u}_h$ vanishes. Finally, since $\boldsymbol{\omega}_h$ and $\boldsymbol{u}_h$ vanishes from (13) and (14) we find that the Lagrange multipliers $\boldsymbol{\lambda}_h$ and p_h must vanish as well. □

In the uniform degree case, our hybridized mixed method is equivalent to the mixed method considered in [14] in the following sense: Our $\boldsymbol{\omega}_h$ and $\boldsymbol{u}_h$ coincide with vorticity and velocity approximations discussed there. Therefore, the error estimates proven there apply to our solution components $\boldsymbol{\omega}_h$ and $\boldsymbol{u}_h$. Note that for the mixed method to make sense, the space $\mathcal{V}_h(0)$ must be non-empty (e.g., in (18)). We tacitly assume that it is. It may appear at this point that our method has too many unknowns. But as we shall see in the next section, it is possible to eliminate all but the Lagrange multiplier variables from (13)–(16), thus making our formulation more attractive.

Before proceeding to the above mentioned elimination, let us note one advantage that results from hybridization: Since hybridization provides an approximation to the pressure on the mesh faces through the Lagrange multiplier p_h, we can compute an approximation to the pressure inside mesh elements in a completely local (element by element) fashion. Borrowing an idea from [4], we define the pressure π_h on the triangle K as the element of $P_{k(K)}(K)$ such that

$$-(\pi_h, \operatorname{div} \boldsymbol{v})_K = (\boldsymbol{f}, \boldsymbol{v})_K - (\mathbf{curl}\, \boldsymbol{\omega}_h, \boldsymbol{v})_K - (\boldsymbol{v} \cdot \boldsymbol{n}, p_h)_{\partial K}, \tag{20}$$

for all $\boldsymbol{v}$ in $P_{k(K)}(K)^3 + \boldsymbol{x}\, P_{k(K)}(K)$, where $\boldsymbol{n}$ denotes the outward unit normal to K. That (20) uniquely defines π_h follows from two facts: (i) div : $P_{k(K)}(K)^3 + \boldsymbol{x}\, P_{k(K)}(K) \mapsto P_{k(K)}(K)$ is a surjection, and (ii) if $\operatorname{div} \boldsymbol{v} = 0$ for a $\boldsymbol{v}$ in $P_{k(K)}(K)^3 + \boldsymbol{x}\, P_{k(K)}(K)$, then $\boldsymbol{v} \in P_{k(K)}(K)^3$ and the right hand side of (20) is zero by the definition of the hybridized method. Thus our method can simultaneously provide approximations to the velocity, vorticity, and pressure.

3. A characterization of the Lagrange multipliers

3.1. **The Lagrange multiplier equation.** We now show how one can eliminate the vorticity as well as the velocity variables from our hybridized mixed method (13)–(16) and arrive at a system of equations involving the Lagrange multipliers alone. Our arguments here are a straightforward generalization of the arguments in [7].

We define *lifting* maps that map functions defined on element interfaces into functions on Ω: Define $(\boldsymbol{w}(\boldsymbol{\lambda}), \boldsymbol{u}(\boldsymbol{\lambda})) \in W_h \times V_h$ and $(\mathbf{w}(p), \mathbf{u}(p)) \in W_h \times V_h$ element by element as follows:

$$(\boldsymbol{w}(\boldsymbol{\lambda}),\boldsymbol{\tau})_K-(\boldsymbol{u}(\boldsymbol{\lambda}),\mathbf{curl}\,\boldsymbol{\tau})_K=(\boldsymbol{\lambda},\boldsymbol{n}\times\boldsymbol{\tau})_{\partial K},\qquad \text{for all } \boldsymbol{\tau}\in W(K), \tag{21}$$
$$(\boldsymbol{v},\mathbf{curl}\;\boldsymbol{w}(\boldsymbol{\lambda}))_K=0,\qquad \text{for all } \boldsymbol{v}\in V(K), \tag{22}$$
$$(\mathsf{w}(p),\boldsymbol{\tau})_K-(\mathbf{u}(p),\mathbf{curl}\,\boldsymbol{\tau})_K=0,\qquad \text{for all } \boldsymbol{\tau}\in W(K), \tag{23}$$
$$(\boldsymbol{v},\mathbf{curl}\;\mathsf{w}(p))_K=-(p,\boldsymbol{v}\cdot\boldsymbol{n})_{\partial K},\qquad \text{for all } \boldsymbol{v}\in V(K). \tag{24}$$

In addition, define $(\mathbf{w}(\boldsymbol{f}),\mathbf{u}(\boldsymbol{f}))$ and $(\boldsymbol{w}(\boldsymbol{g}_{\mathsf{T}}),\boldsymbol{u}(\boldsymbol{g}_{\mathsf{T}}))$ in $W_h\times V_h$ by

$$(\mathbf{w}(\boldsymbol{f}),\boldsymbol{\tau})_K-(\mathbf{u}(\boldsymbol{f}),\mathbf{curl}\,\boldsymbol{\tau})_K=0,\qquad \text{for all } \boldsymbol{\tau}\in W(K), \tag{25}$$
$$(\boldsymbol{v},\mathbf{curl}\;\mathbf{w}(\boldsymbol{f}))_K=(\boldsymbol{f},\boldsymbol{v})_K,\qquad \text{for all } \boldsymbol{v}\in V(K), \tag{26}$$
$$(\boldsymbol{w}(\boldsymbol{g}_{\mathsf{T}}),\boldsymbol{\tau})_K-(\boldsymbol{u}(\boldsymbol{g}_{\mathsf{T}}),\mathbf{curl}\,\boldsymbol{\tau})_K=(\boldsymbol{g}_{\mathsf{T}},\boldsymbol{n}\times\boldsymbol{\tau})_{\partial K\cap\partial\Omega},\qquad \text{for all } \boldsymbol{\tau}\in W(K), \tag{27}$$
$$(\boldsymbol{v},\mathbf{curl}\;\boldsymbol{w}(\boldsymbol{g}_{\mathsf{T}}))_K=0,\qquad \text{for all } \boldsymbol{v}\in V(K). \tag{28}$$

Note that all of the above local problems are uniquely solvable. Hence, these local maps are well defined.

The main result of this section characterizes the Lagrange multipliers as the unique solution of a variational equation involving the bilinear forms,

$$a(\boldsymbol{\lambda},\boldsymbol{\mu})=(\boldsymbol{w}(\boldsymbol{\lambda}),\boldsymbol{w}(\boldsymbol{\mu}))_\Omega,$$
$$c(p,q)=(\mathsf{w}(p),\mathsf{w}(q))_\Omega,$$
$$b(\boldsymbol{\mu},p)=-\sum_{K\in\mathcal{T}}(\boldsymbol{u}(\boldsymbol{\mu}),\mathbf{curl}\;\mathsf{w}(p))_K,$$

and the functionals

$$\ell_1(\boldsymbol{\mu})=(\boldsymbol{f},\boldsymbol{u}(\boldsymbol{\mu}))_\Omega-(\boldsymbol{g}_{\mathsf{T}},\boldsymbol{w}(\boldsymbol{\mu}))_{\partial\Omega} \tag{29}$$
$$\ell_2(q)=(\boldsymbol{f},\mathbf{u}(q))_\Omega+(g_n,q)_{\partial\Omega}-(\boldsymbol{g}_{\mathsf{T}},\mathsf{w}(q))_{\partial\Omega}. \tag{30}$$

Theorem 3.1. *The Lagrange multiplier* $(\boldsymbol{\lambda}_h,p_h)\in M_h\times P_h$ *of the hybridized mixed method* (13)–(16) *is the unique solution of*

$$a(\boldsymbol{\lambda}_h,\boldsymbol{\mu})+b(\boldsymbol{\mu},p_h)=\ell_1(\boldsymbol{\mu}),\qquad \textit{for all } \boldsymbol{\mu}\in M_h \textit{ and} \tag{31}$$
$$b(\boldsymbol{\lambda}_h,q)-c(p_h,q)=\ell_2(q),\qquad \textit{for all } q\in P_h. \tag{32}$$

Moreover, the solution components $\boldsymbol{\omega}_h$ *and* $\boldsymbol{u}_h$ *of the hybridized mixed method* (13)–(16) *can be determined locally as follows:*

$$\boldsymbol{\omega}_h=\boldsymbol{w}(\boldsymbol{\lambda}_h)+\mathsf{w}(p_h)+\boldsymbol{w}(g_t)+\mathbf{w}(\boldsymbol{f}), \tag{33}$$
$$\boldsymbol{u}_h=\boldsymbol{u}(\boldsymbol{\lambda}_h)+\mathbf{u}(p_h)+\boldsymbol{u}(g_t)+\mathbf{u}(\boldsymbol{f}). \tag{34}$$

The proof of this theorem proceeds exactly along the lines of the proof of the analogous theorem in [7].

4. Local bases for Lagrange multipliers

It is clear from Section 3 that one should, in practice, implement our hybridized mixed method not in its direct form (13)–(16), but rather in the reduced form (31)–(32). This requires a computable basis for the Lagrange multiplier spaces M_h and P_h. Local bases for W_h and V_h are obvious as they do not have continuity constraints across mesh faces. But bases for the Lagrange multiplier spaces are not immediate from their definition, so we develop local bases for P_h and M_h in this section. Note that the construction of the basis for the space of tangential velocities in three space dimensions has important differences compared to the two dimensional case.

4.1. **The space of interface pressures.** We begin with a characterization of the space of pressure Lagrange multipliers arising from the first hybridization. To state it, define

$$k(F) = \max\{k(K) : K \in \mathcal{T} \text{ and } K \text{ has } F \text{ as a face}\}, \tag{35}$$

for every $F \in \mathcal{F}$, and set $P(F)$ equal to the space of polynomials of degree at most $k(F)$ on the face F.

Proposition 4.1. *The space P_h defined in* (11) *is characterized by*

$$P_h = \left\{ p : p|_F \in P(F) \text{ for all } F \in \mathcal{F} \text{ and } \sum_{F\in\mathcal{F}} (p,1)_F = 0 \right\}.$$

Note that the use of variable degree spaces requires the pressure Lagrange multiplier to have the *maximum* of the degrees from adjacent elements.

The proof of this proposition is quite similar to that of the two dimensional case considered in [7]. The two main steps of the proof are as follows. In the first, one constructs a local extension $\widetilde{\boldsymbol{v}}_h$ of any given $p \in P_h$ into the Raviart-Thomas space

$$R_h = \{\boldsymbol{r} : \boldsymbol{r}|_K = \boldsymbol{x}p(\boldsymbol{x}) + \boldsymbol{q} \text{ for some } p \in P_{k(K)}(K) \text{ and } \boldsymbol{q} \in P_{k(K)}(K)^3\}$$

such that $[\![\widetilde{\boldsymbol{v}}_h \cdot \boldsymbol{n}]\!] = p$. In the second, one uses a global correction $\boldsymbol{z}_h \in R_h \cap H_0(\mathrm{div}, \Omega)$ such that $\boldsymbol{v}_h = \widetilde{\boldsymbol{v}}_h - \boldsymbol{z}_h$ is in V_h and satisfies $[\![\boldsymbol{v}_h \cdot \boldsymbol{n}]\!] = p$. This is possible by the surjectivity of the divergence map

$$\mathrm{div} : R_h \cap H_0(\mathrm{div}, \Omega) \mapsto S_h,$$

where $S_h = \{v : v|_K \in P_{k(K)}(K)$ and average of v on Ω is zero $\}$. While this surjectivity is a well known property for uniform degree spaces, for the variable degree Raviart-Thomas space, it follows from our results in [6]. The remaining details of the proof of Proposition 4.1 are identical to its two dimensional analogue in [7], so we omit them.

By Proposition 4.1, the Lagrange multiplier space P_h can be identified with $\widetilde{P}_h/\mathbb{R}$ where

$$\widetilde{P}_h = \{p : p|_F \in P_k(F) \text{ for all } F \in \mathcal{F}\}.$$

Obviously, we can construct a basis for $\widetilde{P}_h$ by taking the union of local bases for $P_k(F)$, say Legendre polynomials, on every edge $F \in \mathcal{F}$. It is enough to construct such a basis in computations.

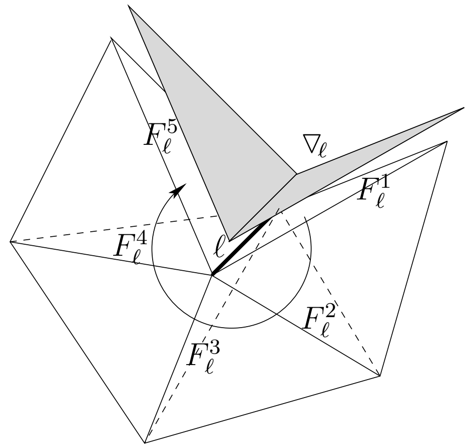

FIGURE 1. Construction of basis functions supported near a mesh edge ℓ

4.2. **The lowest order tangential velocity space.** Now, we begin the construction of a local basis for the space M_h of tangential velocity Lagrange multipliers. In this subsection, we study the lowest order case which is easier to describe. In the next subsection, we consider the general case.

In order to explicitly give a local basis for M_h, we introduce some more notation. Let K be a tetrahedron in $\mathcal{T}$ and ℓ be one of its edges. We denote by $\Lambda_{\ell,K}$ the union of the two faces of K that share the edge ℓ. Define the collection of such *wedges* by

$$\hat{\Lambda}_h = \{\Lambda_{\ell,K} : \ell \text{ is an edge of } \mathcal{T} \text{ and } K \in \mathcal{T}\}.$$

For all $\Lambda \in \hat{\Lambda}_h$, we denote by K_Λ the (unique) tetrahedron $K \in \mathcal{T}$ such that $\Lambda \subseteq \partial K$. The edge of a wedge Λ is the common edge of the two faces that form Λ. This edge is denoted by ℓ_Λ. Let β_i and β_j be the barycentric coordinate functions (with respect to the tetrahedron K_Λ) associated with the two endpoints of ℓ_Λ . Set

$$\boldsymbol{\phi}_\Lambda = \begin{cases} \beta_i \boldsymbol{\nabla} \beta_j - \beta_j \boldsymbol{\nabla} \beta_i, & \text{on } K_\Lambda, \\ 0, & \text{on all other } K \in \mathcal{T}. \end{cases}$$

We define a basis for M_h using the functions

$$\boldsymbol{\psi}_\Lambda = [\![\boldsymbol{n} \times \boldsymbol{\phi}_\Lambda]\!].$$

Since $\boldsymbol{\phi}_\Lambda \in W_h$, the functions $\boldsymbol{\psi}_\Lambda$ are in M_h by definition. But not all of $\boldsymbol{\psi}_\Lambda, \Lambda \in \hat{\Lambda}_h$ are linearly independent, e.g., the functions $\boldsymbol{\psi}_\Lambda$ for all Λ connected to one edge are linked by one equation. Therefore, for every mesh edge ℓ (including edges $\ell \subset \partial\Omega$), we arbitrarily pick one wedge $\Lambda \in \hat{\Lambda}_h$ with edge $\ell_\Lambda = \ell$, denote it by $\triangledown_\ell$ (see Fig. 1), and "omit" it: Define

$$\Lambda_h = \hat{\Lambda}_h \setminus \{\triangledown_\ell : \text{ for all mesh edges } \ell\}.$$

Proposition 4.2. *The set* $\mathcal{B}_0 = \{\boldsymbol{\psi}_\Lambda : \Lambda \in \Lambda_h\}$ *is a basis for* M_h *whenever* $k(K) = 0$ *for all* $K \in \mathcal{T}$.

Proof. Since the span of $\mathcal{B}_0$ is contained in M_h, it suffices to prove that

$$\operatorname{card}\mathcal{B}_0 = \dim M_h, \quad \text{and} \tag{36}$$

$$\mathcal{B}_0 \text{ is a linearly independent set.} \tag{37}$$

To prove (36), let us first count the dimension of M_h. Defining $T_h : W_h \mapsto M_h$ by

$$T_h\boldsymbol{\tau} = [\![\boldsymbol{n}\times\boldsymbol{\tau}]\!],$$

we note that M_h is the range of T_h. Since the null space of T_h is $\mathcal{W}_h$, by the rank-nullity theorem, we find that

$$\dim(M_h) = \operatorname{rank}(T_h) = \dim(W_h) - \dim(\mathcal{W}_h). \tag{38}$$

Now, $W(K)$ in the lowest order case is a space of dimension six. Since the number of degrees of freedom of the conforming lowest order Nédélec space $\mathcal{W}_h$ equals the number of edges n_E in the mesh, we find that

$$\dim(M_h) = 6n_K - n_E,$$

where n_K is the number of tetrahedra in the mesh $\mathcal{T}$. Thus

$$\operatorname{card}\mathcal{B}_0 = \operatorname{card}\varLambda_h = \operatorname{card}\hat{\varLambda}_h - n_E = 6n_K - n_E,$$

which coincides with $\dim M_h$.

Now, let us prove (37). We want to show that if

$$\boldsymbol{\mu} = \sum_{\varLambda\in\varLambda_h} c_\varLambda \boldsymbol{\psi}_\varLambda \tag{39}$$

vanishes, then all the coefficients $c_\varLambda$ are zero. Notice that the function $\boldsymbol{\mu}$, in general, is not well defined at the edge ℓ, as the limits of $\boldsymbol{\mu}$ from various faces sharing the edge ℓ can differ. In order to examine these limits, we introduce the following notation. Enumerate all $\varLambda \in \varLambda_h$ with edge ℓ as $\varLambda_\ell^1, \varLambda_\ell^2, \ldots, \varLambda_\ell^{N_\ell}$ and all faces in $\mathcal{F}$ sharing the edge ℓ as $F_\ell^1, F_\ell^2, \ldots F_\ell^{N_\ell+1}$ (see Fig. 1) in such a way that the two faces of $\varLambda_\ell^j$ are F_ℓ^j and F_ℓ^{j+1} and the two faces of $\triangledown_\ell$ are F_ℓ^1 and $F_\ell^{N_\ell+1}$. Let $\boldsymbol{t}_F$ be the unit tangent vector along ∂F fixed by arbitrarily choosing one of the two possible orientations. Let $\boldsymbol{n}_F$ be a unit vector normal to F chosen by the right hand rule and

$$\boldsymbol{\nu}_F = \boldsymbol{t}_F \times \boldsymbol{n}_F. \tag{40}$$

Note that both the choices of orientation for $\boldsymbol{t}_F$ yield the same $\boldsymbol{\nu}_F$, which represents the outward unit normal of F relative to the plane containing F.

Our proof proceeds by examining the following functions on the edge ℓ:

$$\mu_\ell^i := \left(\boldsymbol{\mu}|_{F_\ell^i}\right)\cdot \boldsymbol{\nu}_{F_\ell^i}\Big|_\ell = \sum_{\varLambda\in\varLambda_h} c_\varLambda\left(\boldsymbol{\psi}_\varLambda|_{F_\ell^i}\cdot\boldsymbol{\nu}_{F_\ell^i}\right)\Big|_\ell.$$

Now, there are at most five $\varLambda \in \varLambda_h$ such that $\boldsymbol{\psi}_\varLambda$ is nonzero on the face F_ℓ^1. Moreover, only one of them has nonzero normal trace $\boldsymbol{\psi}_\varLambda\cdot\boldsymbol{\nu}_{F_\ell^i}$ on ℓ, namely $\boldsymbol{\psi}_{\varLambda_\ell^1}$. Hence

$$\mu_\ell^1 = c_{\varLambda_\ell^1}\left(\boldsymbol{\psi}_{\varLambda_\ell^1}|_{F_\ell^i}\cdot\boldsymbol{\nu}_{F_\ell^1}\right)\Big|_\ell.$$

It then follows that

$$|\mu_\ell^1| = \left|\boldsymbol{\nu}_{F_\ell^1}\cdot(c_{\varLambda_\ell^1}\boldsymbol{\psi}_{\varLambda_\ell^1})|_\ell\right| = \left|c_{\varLambda_\ell^1}\,(\boldsymbol{\nu}_{F_\ell^1}\times\boldsymbol{n}_{F_\ell^1})\cdot\boldsymbol{\phi}_{\varLambda_\ell^1}|_\ell\right| = \left|c_{\varLambda_\ell^1}\,(\boldsymbol{t}_{F_\ell^1}\cdot\boldsymbol{\phi}_{\varLambda_\ell^1})|_\ell\right| = \frac{1}{h_\ell}|c_{\varLambda_\ell^1}|,$$

where h_ℓ denotes the length of the edge ℓ. Similarly, we also find that $|\mu_\ell^{N_\ell+1}| = |c_\Lambda^{N_\ell+1}|$ and

$$|\mu_\ell^j| = \frac{1}{h_\ell}|c_{\Lambda_x^j} - c_{\Lambda_x^{j-1}}| \qquad \text{for all } j = 2,\dots,N_\ell.$$

If $\boldsymbol{\mu}$ vanishes everywhere, then for any mesh edge ℓ, the function μ_ℓ^j defined above must vanish on the edge ℓ. Hence

$$\begin{aligned} |c_{\Lambda_\ell^1}| = |c_{\Lambda_\ell^{N_\ell+1}}| &= 0, && \text{and} \\ |c_{\Lambda_\ell^j} - c_{\Lambda_\ell^{j+1}}| &= 0, && \text{for all } j = 2,\dots,N_\ell. \end{aligned}$$

Hence $c_{\Lambda_\ell^j} = 0$ for all j. This argument applies to every mesh edge, so all the coefficients c_Λ in (39) are zero. Hence (37) follows. □

4.3. **The higher order space of tangential velocities.** In this subsection we show how to construct a local basis for the Lagrange multiplier space M_h in the general case of the higher order spaces and the variable degree method. Here there is one important difference with the two dimensional case. In the two dimensional case [7], we were able to obtain a basis for the higher order space by augmenting the lowest order basis with some edge basis functions. In the three dimensional case however, we cannot expect to get a basis for the higher order space by just augmenting $\mathcal{B}_0$ with some face basis functions. This is because while in two dimensions, a vertex represents at most one degree of freedom, in three dimensions, an edge can have more than one degree of freedom associated to it. Thus we must add to $\mathcal{B}_0$ functions that represent face degrees of freedom as well as functions that represent the additional edge degrees of freedom.

In order to give a basis explicitly, as well as to understand the nature of our space of tangential velocities, it is convenient to recall a basis for the Nédélec space given in [10]. For any integer $k \geq 0$ and any N-simplex D ($N = 2$ or 3 for our purposes), the Nédélec space is

$$W_k(D) = P_k(D)^N \oplus S_{k+1}(D).$$

Let $\beta_1,\dots,\beta_{N+1}$ denote the $N+1$ barycentric coordinate functions of the N-simplex D. Let $I_{lm}(N,k)$ denote the set of all multi-indices $\boldsymbol{\alpha} := (\alpha_1,\dots,\alpha_N)$ (where α_i are non-negative integers) such that $\alpha_i = 0$ for all i not equal to l or m and $\alpha_l + \alpha_m = k$. Similarly, $I_{lmn}(N,k)$ is the set of multi-indices $\boldsymbol{\alpha}$ with $\alpha_i = 0$ for all i not equal to l, m, or n, and $\alpha_l + \alpha_m + \alpha_n = k$. Using powers of barycentric coordinates (for $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_{N+1})$, we define $\boldsymbol{\beta}^{\boldsymbol{\alpha}} := \beta_1^{\alpha_1}\dots\beta_{N+1}^{\alpha_{N+1}}$) we introduce the following sets of functions:

$$\mathcal{B}_{lm}^{(D)} = \bigcup_{\boldsymbol{\alpha}\in I_{lm}(N+1,k)} \Big\{\boldsymbol{\beta}^{\boldsymbol{\alpha}}(\beta_l \,\boldsymbol{\nabla}\,\beta_m - \beta_m \,\boldsymbol{\nabla}\,\beta_l)\Big\}, \tag{41}$$

$$\mathcal{B}_{lmn}^{(D)} = \bigcup_{\boldsymbol{\alpha}\in I_{lmn}(N+1,k-1)} \Big\{\boldsymbol{\beta}^{\boldsymbol{\alpha}}(\beta_l\beta_m \,\boldsymbol{\nabla}\,\beta_n - \beta_m\beta_n \,\boldsymbol{\nabla}\,\beta_l),\ \boldsymbol{\beta}^{\boldsymbol{\alpha}}(\beta_m\beta_n \,\boldsymbol{\nabla}\,\beta_l - \beta_n\beta_l \,\boldsymbol{\nabla}\,\beta_m)\Big\}. \tag{42}$$

From the results of [10], it now follows that if D is a triangle, then the union of the sets $\mathcal{B}_{12}^{(D)}$, $\mathcal{B}_{23}^{(D)}$, $\mathcal{B}_{31}^{(D)}$ and $\mathcal{B}_{123}^{(D)}$, form a basis for the Nédélec space $W_k(D)$. If D is a tetrahedron instead, then a basis for $W_k(D)$ is

$$\mathcal{B}_{12}^{(D)} \cup \mathcal{B}_{13}^{(D)} \cup \mathcal{B}_{14}^{(D)} \cup \mathcal{B}_{23}^{(D)} \cup \mathcal{B}_{24}^{(D)} \cup \mathcal{B}_{34}^{(D)} \cup \mathcal{B}_{123}^{(D)} \cup \mathcal{B}_{124}^{(D)} \cup \mathcal{B}_{134}^{(D)} \cup \mathcal{B}_{234}^{(D)} \cup \mathcal{B}_{1234}^{(D)},$$

where

$$\mathcal{B}_{1234}^{(D)} := \bigcup \{ \boldsymbol{\beta}^{\boldsymbol{\alpha}} (\beta_1\beta_2\beta_3 \boldsymbol{\nabla} \beta_4 - \beta_2\beta_3\beta_4 \boldsymbol{\nabla} \beta_1), \boldsymbol{\beta}^{\boldsymbol{\alpha}} (\beta_2\beta_3\beta_4 \boldsymbol{\nabla} \beta_1 - \beta_3\beta_4\beta_1 \boldsymbol{\nabla} \beta_2),$$
$$\boldsymbol{\beta}^{\boldsymbol{\alpha}} (\beta_3\beta_4\beta_1 \boldsymbol{\nabla} \beta_2 - \beta_4\beta_3\beta_2 \boldsymbol{\nabla} \beta_1) : \boldsymbol{\alpha} \in I_{1234}(N+1, k-2) \}.$$

Note that the basis functions in (41) are "edge" basis functions, those in (42) are "face" basis functions and the ones in $\mathcal{B}_{1234}^{(D)}$ are "interior" basis functions, in the sense explained in [10].

Since the Lagrange multiplier space M_h is obtained using the tangential traces of functions in W_h, it is instructive to study the space of tangential traces of the Nédélec space on one tetrahedron K. Let $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ and $\boldsymbol{a}_4$ be the vertices of K, F_{lmn} be the face formed by $\boldsymbol{a}_l, \boldsymbol{a}_m$ and $\boldsymbol{a}_n$, and e_{ij} be the edge formed by $\boldsymbol{a}_i$ and $\boldsymbol{a}_j$. We denote by $\boldsymbol{n} \times W_k(K)$ the space of functions on ∂K of the form $\boldsymbol{n} \times \boldsymbol{w}$ for some $\boldsymbol{w} \in W_k(K)$. Recall that for any N-dimensional domain D, the Raviart-Thomas space of polynomials is $R_k(D) = \boldsymbol{x} P_k(D) + P_k(D)^N$ where $\boldsymbol{x}$ is the coordinate vector on D. Define the Raviart-Thomas space on the manifold ∂K by

$$R_k(\partial K) = \{ \boldsymbol{r} : \ \boldsymbol{r}|_{F_{ijl}} \in R_k(F_{ijl}) \text{ and } (\boldsymbol{r}|_{F_{ijl}}) \cdot \boldsymbol{\nu}_{F_{ijl}} + (\boldsymbol{r}|_{F_{ijm}}) \cdot \boldsymbol{\nu}_{F_{ijm}} = 0 \text{ on } e_{ij}, \text{ for all } i, j, l, m \},$$

where we have used the notation in (40). Then we have the following result:

Proposition 4.3. *The space of tangential traces of the Nédélec space $\boldsymbol{n} \times W_k(K)$ is the Raviart-Thomas space $R_k(\partial K)$.*

Proof. We begin by proving that $\boldsymbol{n} \times W_k(K) \subseteq R_k(\partial K)$. Let the tangential component of $\boldsymbol{w} \in W_k(K)$ on ∂K be denoted by $\boldsymbol{w}_{\mathsf{T}}$. We first prove that $\boldsymbol{w}_{\mathsf{T}}$ on face F_{lmn} is in $R_k(F_{lmn})$. It is easy to see from the structure of the basis functions in (41) that if $\boldsymbol{w} \in \mathcal{B}_{ij}^{(K)}$ then $\boldsymbol{w}_{\mathsf{T}}|_{F_{lmn}}$ is zero if i or j does not belong to $\{l, m, n\}$. If both i and j are in $\{l, m, n\}$ then $\boldsymbol{w}_{\mathsf{T}}|_{F_{lmn}} \in \mathcal{B}_{ij}^{(F_{lmn})}$. Therefore, we find that $\boldsymbol{w}_{\mathsf{T}}|_{F_{lmn}}$ is in the Nédélec space $W_k(F_{lmn})$.

In two dimensions, the Nédélec space is the "rotated" Raviart-Thomas space. Indeed, if D is a triangle in the x-y plane then

$$W_k(D) = P_k(K)^2 \oplus S_{k+1}(K) = P_k(K)^2 \oplus \begin{pmatrix} -y \\ x \end{pmatrix} P_k(K).$$

Since $\boldsymbol{n} \times \boldsymbol{w}|_{F_{lmn}}$ is $\boldsymbol{w}_{\mathsf{T}}|_{F_{lmn}}$ rotated (by an angle of $\pi/2$), it follows that the tangential trace $\boldsymbol{n} \times \boldsymbol{w}$ on F_{lmn} is in the Raviart-Thomas space $R_k(F_{lmn})$.

To show the continuity of the normal components of $\boldsymbol{n} \times \boldsymbol{w}$ across edges of ∂K, consider an edge e_{ij} shared by two faces F_{ijl} and F_{ijm}. Then by (40) and the continuity of the tangential components of the Nédélec space, we have the following equalities on e_{ij}:

$$(\boldsymbol{n} \times \boldsymbol{w})|_{F_{ijl}} \cdot \boldsymbol{\nu}_{F_{ijl}} = (\boldsymbol{\nu}_{F_{ijl}} \times \boldsymbol{n}) \cdot \boldsymbol{w}|_{Fijl} = -\boldsymbol{t}_{F_{ijl}} \cdot \boldsymbol{w}|_{Fijl}$$
$$= \boldsymbol{t}_{F_{ijm}} \cdot \boldsymbol{w}|_{Fijm} = -(\boldsymbol{n} \times \boldsymbol{w})|_{Fijm} \cdot \boldsymbol{\nu}_{F_{ijm}}.$$

Thus $\boldsymbol{n} \times \boldsymbol{w}$ is in $R_k(\partial K)$ whenever $\boldsymbol{w} \in W_k(K)$. It is easy to see that all functions in $R_k(\partial K)$ can be obtained as tangential traces of $W_k(K)$. □

Now we are ready to describe the building blocks of a basis for the general higher order M_h arising from the variable degree Nédélec spaces. The basis is divided into two parts: one corresponding to the interior faces of the mesh and another corresponding to the wedges in

$\varLambda_h$. The former is easy to describe: Let $\mathcal{F}_0$ denote the set of all interior faces of the mesh $\mathcal{T}$. For any face $F \in \mathcal{F}_0$, define

$$\mathring{V}(F) = \{\boldsymbol{w} \in R_{k(F)}(F) : \boldsymbol{w}|_{\partial F} \cdot \boldsymbol{\nu}_F = 0 \text{ on } \partial F\},$$

where $k(F)$ is the *maximum* of the degrees from either side of F, as defined in (35). Let $\mathring{\mathcal{B}}_F$ be a basis for $\mathring{V}(F)$.

To describe the wedge basis functions, recall the notations introduced in the previous subsection. Now we additionally require that for every mesh edge ℓ, the "omitted wedge" $\triangledown_\ell$ is associated to a tetrahedron (having ℓ as an edge and) having the *minimal* degree: More precisely, we choose $\triangledown_\ell$ such that

$$k(K_{\triangledown_\ell}) = \min_{i=1,\ldots,N_\ell} k(K_{\varLambda_\ell^i}). \tag{43}$$

For all the remaining $\varLambda \in \varLambda_h$, we define the following Raviart-Thomas type space:

$$R(\varLambda) = \{\boldsymbol{r} \in R_{k(K_\varLambda)}(\partial K_\varLambda) : \ \boldsymbol{r} \text{ is supported on } \varLambda\}.$$

Just as we decompose the standard Raviart-Thomas space, we can decompose $R(\varLambda)$ into subspaces corresponding to interior and boundary degrees of freedom: If $F_\varLambda^+$ and $F_\varLambda^-$ denote the two faces of $\varLambda$ and $\mathring{R}(F_\varLambda^\pm) = \{\boldsymbol{r} \in R_{k(K_\varLambda)}(\partial K_\varLambda) : \ \boldsymbol{r} \text{ is supported on } F_\varLambda^\pm\}$, we can decompose $R(\varLambda) = \mathring{R}(F_\varLambda^+) \oplus \mathring{R}(F_\varLambda^-) \oplus V(\varLambda)$ where $V(\varLambda)$ is a subspace that is linearly independent to $\mathring{R}(F_\varLambda^+) \oplus \mathring{R}(F_\varLambda^-)$, e.g., we can choose $V(\varLambda)$ to be the $L^2(\varLambda)$-orthogonal complement of $\mathring{R}(F_\varLambda^+) \oplus \mathring{R}(F_\varLambda^-)$ in $R(\varLambda)$. (Another alternative is suggested in the next paragraph.) Let $\mathcal{B}_\varLambda$ be a basis for $V(\varLambda)$. Our next theorem shows that such wedge basis functions together with the face basis functions form a basis for the global space M_h.

Particular examples of $\mathcal{B}_\varLambda$ and $\mathring{\mathcal{B}}_F$ are easy to exhibit. We give one conveniently implementable choice that follow from the previous results of [10]. Let $\varLambda \in \varLambda_h$ and let β_i, $i = 1, 2, 3, 4$ denote the barycentric coordinates of $K_\varLambda$ such that β_i and β_j are associated to the two endpoints of the edge $\ell_\varLambda$. Define

$$\boldsymbol{\phi}_\varLambda^{(\boldsymbol{\alpha})} = \begin{cases} \boldsymbol{\beta}^{\boldsymbol{\alpha}}(\beta_i \boldsymbol{\nabla} \beta_j - \beta_j \boldsymbol{\nabla} \beta_i) & \text{on } K_\varLambda \\ 0 & \text{on all other } K \in \mathcal{T}. \end{cases}$$

for all $\boldsymbol{\alpha} \in I_{ij}(4, k(K_\varLambda))$ and

$$\boldsymbol{\psi}_\varLambda^{(\boldsymbol{\alpha})} = [\![\boldsymbol{n} \times \boldsymbol{\phi}_\varLambda^{(\boldsymbol{\alpha})}]\!].$$

We can choose

$$\mathcal{B}_\varLambda = \{\boldsymbol{\psi}_\varLambda^{(\boldsymbol{\alpha})} : \boldsymbol{\alpha} \in I_{ij}(4, k(K_\varLambda))\}.$$

For an example of a face basis, let $F \in \mathcal{F}_0$. If β_i, β_j and β_k are the three barycentric coordinate functions of the face F, then we may choose

$$\mathring{\mathcal{B}}_F = \bigcup_{\boldsymbol{\alpha} \in I_{ijk}(3, k(F)-1)} \left\{ \boldsymbol{\beta}^{\boldsymbol{\alpha}}(\beta_i\beta_j \boldsymbol{\nabla} \beta_k - \beta_j\beta_k \boldsymbol{\nabla} \beta_i) \times \boldsymbol{n}_F, \ \boldsymbol{\beta}^{\boldsymbol{\alpha}}(\beta_j\beta_k \boldsymbol{\nabla} \beta_i - \beta_k\beta_i \boldsymbol{\nabla} \beta_j) \times \boldsymbol{n}_F \right\}. \tag{44}$$

The following theorem gives a basis for M_h.

Theorem 4.1. *The set*

$$\mathcal{B} = \left(\bigcup_{\varLambda \in \varLambda_h} \mathcal{B}_\varLambda \right) \cup \left(\bigcup_{F \in \mathcal{F}_0} \mathring{\mathcal{B}}_F \right)$$

is a basis for M_h.

Proof. It follows from Proposition 4.3 that elements of $\mathring{\mathcal{B}}_F$ and $\mathcal{B}_\Lambda$ can be written as $[\![\boldsymbol{n}\times\boldsymbol{\phi}]\!]$ for some $\boldsymbol{\phi}\in W_h$. Hence the span of $\mathcal{B}$ is contained in M_h. It now suffices to prove that

$$\operatorname{card}\mathcal{B} = \dim(M_h), \tag{45}$$

and that $\mathcal{B}$ is a linearly independent set. For any $\boldsymbol{\mu}\in\mathring{\mathcal{B}}_F$, the normal trace from F on ∂F vanishes:

$$(\boldsymbol{\mu}|_{\partial F})\cdot\boldsymbol{\nu}_F = 0.$$

The normal traces of functions in $\mathcal{B}_\Lambda$ from Λ on ℓ_Λ are linearly independent. Hence by a minor modification of the arguments in the proof of Proposition 4.2, the linear independence of $\mathcal{B}$ follows from the linear independence of functions within $\mathcal{B}_\Lambda$ and $\mathring{\mathcal{B}}_F$.

To prove (45), let us first count the number of elements in $\mathcal{B}$. The dimension of $\mathring{V}(F)$ can be calculated easily (either directly or using (44)). It equals

$$\operatorname{card}\mathring{\mathcal{B}}_F = 2\operatorname{card} I_{123}(3, k(F)-1) = k(F)\big(k(F)+1\big).$$

Moreover,

$$\operatorname{card}\mathcal{B}_\Lambda = \operatorname{card} I_{12}(4, k(K_\Lambda)) = k(K_\Lambda)+1.$$

Thus,

$$\operatorname{card}\mathcal{B} = \sum_{\Lambda\in\Lambda_h}(k(K_\Lambda)+1) + \sum_{F\in\mathcal{F}_0} k(F)\big(k(F)+1\big). \tag{46}$$

Now let us compute the dimension of M_h by using the following identity (see (38))

$$\dim(M_h) = \dim(W_h) - \dim(\mathcal{W}_h).$$

By the tangential continuity conditions on the variable degree space $\mathcal{W}_h$, we find that the space of traces $\boldsymbol{n}_F\times\boldsymbol{w}$ on a face $F\in\mathcal{F}$ for $\boldsymbol{w}\in\mathcal{W}_h$ is $R_{\underset{\sim}{k}(F)}(F)$ where

$$\underset{\sim}{k}(F) = \min\{k(K) : K\in\mathcal{T} \text{ and } K \text{ has } F \text{ as a face}\}.$$

Furthermore, the tangential component $\boldsymbol{w}\cdot\boldsymbol{t}$ on an edge E is in $P_{\underset{\sim}{k}(E)}(E)$ where

$$\underset{\sim}{k}(E) = \min\{k(K) : K\in\mathcal{T} \text{ and } K \text{ has } E \text{ as an edge}\}.$$

Splitting the global degrees of freedom of $\mathcal{W}_h$ as edge degrees of freedom, face degrees of freedom, and interior degrees of freedom, we find that

$$\dim(\mathcal{W}_h) = \sum_{E\in\mathcal{E}}\big(\underset{\sim}{k}(E)+1\big) + \sum_{F\in\mathcal{F}}\underset{\sim}{k}(F)\big(\underset{\sim}{k}(F)+1\big) + \sum_{K\in\mathcal{T}}\frac{1}{2}\big(k(K)-1\big)k(K)\big(k(K)+1\big).$$

Consequently,

$$\begin{aligned}\dim(W_h)-\dim(\mathcal{W}_h) = {}&\Big(\sum_{K\in\mathcal{T}} 6\big(k(K)+1\big) - \sum_{E\in\mathcal{E}}\big(\underset{\sim}{k}(E)+1\big)\Big)\\ &+\Big(\sum_{K\in\mathcal{T}} 4k(K)\big(k(K)+1\big) - \sum_{F\in\mathcal{F}}\underset{\sim}{k}(F)\big(\underset{\sim}{k}(F)+1\big)\Big).\end{aligned} \tag{47}$$

Because of (38), it suffices to show that the above equals $\operatorname{card}\mathcal{B}$.

In order to do this, we simplify the right hand side of (47). Observe that by rearrangement,

$$\sum_{K\in\mathcal{T}} 4k(K)\big(k(K)+1\big) = \sum_{F\in\mathcal{F}}\Big(k(K_F^+)\big(k(K_F^+)+1\big) + k(K_F^-)\big(k(K_F^-)+1\big)\Big)$$

where $K_F^{\pm}$ is a defined earlier and one of $k(K_F^{\pm})$ is understood to vanish if $F \subseteq \partial\Omega$. Hence

$$\sum_{K\in\mathcal{T}} 4k(K)\big(k(K)+1\big) - \sum_{F\in\mathcal{F}} \underset{\sim}{k}(F)\big(\underset{\sim}{k}(F)+1\big) = \sum_{F\in\mathcal{F}_0} k(F)\big(k(F)+1\big).$$

Similarly, denoting by K_ℓ^i, $i = 1, 2, \ldots, N_\ell$, the tetrahedra in $\mathcal{T}$ which have ℓ as an edge, the rearrangement

$$\sum_{K\in\mathcal{T}} 6\big(k(K)+1\big) = \sum_{\ell\in\mathcal{E}}\sum_{i=1}^{N_\ell} \big(k(K_\ell^i)+1\big)$$

implies, in view of (43), that

$$\begin{aligned}\sum_{K\in\mathcal{T}} 6\big(k(K)+1\big) - \sum_{E\in\mathcal{E}} \big(\underset{\sim}{k}(E)+1\big) &= \sum_{\ell\in\mathcal{E}} \Big(\sum_{i=1}^{N_\ell} k(K_\ell^i)+1\Big) - \sum_{\ell\in\mathcal{E}} \big(k(K_{\nabla_\ell})+1\big)\\ &= \sum_{\Lambda\in\Lambda_h} \big(k(K_\Lambda)+1\big).\end{aligned}$$

Using these identities in (47) we obtain

$$\dim(W_h) - \dim(\mathcal{W}_h) = \sum_{\Lambda\in\Lambda_h} \big(k(K_\Lambda)+1\big) + \sum_{F\in\mathcal{F}_0} k(F)\big(k(F)+1\big)$$

which coincides with $\operatorname{card}\mathcal{B}$ as computed in (46). Hence (45) follows. □

5. Formulae for the lowest order case

In [7], we discussed a few implementation techniques to implement and solve the two dimensional analogue of the Lagrange multiplier system (31)–(32). The considerations there apply to the three dimensional case as well. In particular, one can form the stiffness matrix of (31)–(32) and then perform one further elimination (of the pressure multiplier) to obtain a Schur complement system involving the tangential velocity variable $\boldsymbol{\lambda}_h$ alone. We do not repeat this and other details discussed in [7]. However, since the formulae for the liftings change in three dimensions, we give here new formulae for the liftings as well as local stiffness matrices for the lowest order case.

First, consider the local maps which define the linear and bilinear forms in (31)–(32) for the lowest order case (i.e., $k(K) = 0$ for all $K \in \mathcal{T}$). Let K be any tetrahedron in $\mathcal{T}$. Simple computations show that

$$\begin{aligned}
\boldsymbol{w}(\boldsymbol{\lambda}) &= \frac{1}{|K|}\int_{\partial K\setminus\partial\Omega} \boldsymbol{\lambda}\times\boldsymbol{n}\,\mathrm{d}s, &\quad \boldsymbol{u}(\boldsymbol{\lambda}) &= \frac{1}{2|K|}\int_{\partial K\setminus\partial\Omega} (\boldsymbol{x}-\boldsymbol{x}_K)\times(\boldsymbol{n}\times\boldsymbol{\lambda})\,\mathrm{d}s,\\
\boldsymbol{w}(\boldsymbol{g}_{\mathsf{T}}) &= \frac{1}{|K|}\int_{\partial K\cap\partial\Omega} \boldsymbol{g}_{\mathsf{T}}\times\boldsymbol{n}\,\mathrm{d}s, &\quad \boldsymbol{u}(\boldsymbol{g}_{\mathsf{T}}) &= \frac{1}{2|K|}\int_{\partial K\cap\partial\Omega} (\boldsymbol{x}-\boldsymbol{x}_K)\times(\boldsymbol{n}\times\boldsymbol{g}_{\mathsf{T}})\,\mathrm{d}s,\\
\mathbf{w}(p) &= \mathbf{w}_p^K\times(\boldsymbol{x}-\boldsymbol{x}_K), &\quad \mathbf{u}(p) &= \frac{1}{2|K|}\int_K (\boldsymbol{x}-\boldsymbol{x}_K)\times\mathbf{w}(p)\,\mathrm{d}x,\\
\mathbf{w}(\boldsymbol{f}) &= \mathbf{w}_{\boldsymbol{f}}^K\times(\boldsymbol{x}-\boldsymbol{x}_K), &\quad \mathbf{u}(\boldsymbol{f}) &= \frac{1}{2|K|}\int_K (\boldsymbol{x}-\boldsymbol{x}_K)\times\mathbf{w}(\boldsymbol{f})\,\mathrm{d}x,
\end{aligned}$$

where the point $\boldsymbol{x}_K$ denotes the barycenter of the tetrahedron K,

$$\mathbf{w}_p^K = -\frac{1}{2|K|}\int_{\partial K} p\,\boldsymbol{n}\,\mathrm{d}s, \qquad \mathbf{w}_{\boldsymbol{f}}^K = \frac{1}{2|K|}\int_K \boldsymbol{f}\,\mathrm{d}x.$$

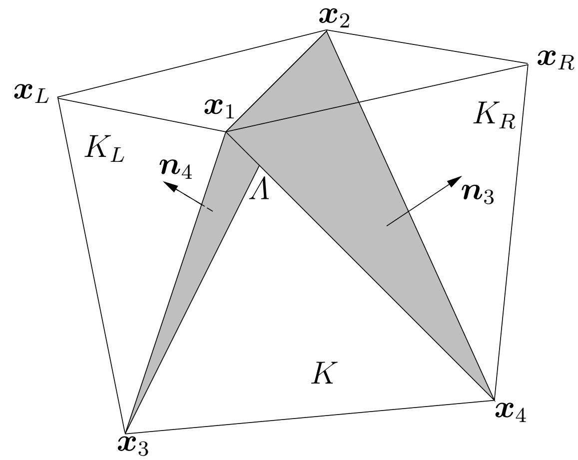

FIGURE 2. The lifting of the basis function from Λ is supported on three mesh tetrahedra K, K_L, and K_R.

Here and elsewhere we use $|X|$ to denote the measure of X.

In order to implement (31)–(32), one uses the basis for M_h and P_h described previously, applies the above local lifting maps to the basis functions, and forms local stiffness matrices of the bilinear forms of $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$. In line with these steps, we next simplify the above expressions in the case of a lowest order basis function of M_h and P_h. Let K be the tetrahedron formed by vertices $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ and $\boldsymbol{x}_4$. Let F_i denote the face of K opposite to vertex x_i, and $\boldsymbol{n}_i$ denote the outward unit normal of K on the face F_i. We first give the lifting of $\boldsymbol{\psi}_\Lambda$, a basis function associated with a $\Lambda \in \Lambda_h$ with $K_\Lambda = K$ (see Figure 2). Since we are considering the lowest order case, by definition,

$$\boldsymbol{\psi}_\Lambda = \begin{cases} \boldsymbol{n}_3 \times (\beta_1 \boldsymbol{\nabla} \beta_2 - \beta_2 \boldsymbol{\nabla} \beta_1) & \text{on face } F_3, \\ \boldsymbol{n}_4 \times (\beta_1 \boldsymbol{\nabla} \beta_2 - \beta_2 \boldsymbol{\nabla} \beta_1) & \text{on face } F_4, \\ 0 & \text{on all other mesh faces.} \end{cases}$$

It is easily seen that the above expression is equal to the following:

$$\boldsymbol{\psi}_\Lambda = \begin{cases} -\dfrac{1}{2|F_3|}(\boldsymbol{x} - \boldsymbol{x}_4) & \text{on face } F_3, \\ -\dfrac{1}{2|F_4|}(\boldsymbol{x} - \boldsymbol{x}_3) & \text{on face } F_4, \\ 0 & \text{on all other mesh faces.} \end{cases}$$

The computations are simplified by working with the latter expression for $\boldsymbol{\psi}_\Lambda$, which also illustrates the connection of the tangential traces with the Raviart-Thomas space. The liftings $\boldsymbol{w}_\Lambda := \boldsymbol{w}(\boldsymbol{\psi}_\Lambda)$ and $\boldsymbol{u}_\Lambda := \boldsymbol{u}(\boldsymbol{\psi}_\Lambda)$ are supported on three tetrahedra, unless Λ intersects $\partial\Omega$. Since the formulae one obtains when Λ intersects $\partial\Omega$ is similar to the remaining cases, we only consider the case shown in Figure 2, where the lifting is supported on the three tetrahedra shown, namely, K, K_L and K_R. Letting $\boldsymbol{x}_{ij} = \boldsymbol{x}_i - \boldsymbol{x}_j$ for any subscripts i and j, we

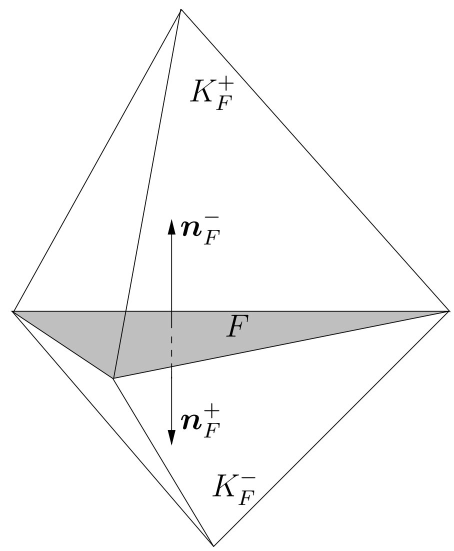

FIGURE 3. The lifting of the pressure basis function from a face F is supported on the tetrahedra adjacent to the face F.

have

$$\begin{aligned}
\boldsymbol{w}_\Lambda &= -\frac{(\boldsymbol{x}_{31}+\boldsymbol{x}_{32})\times \boldsymbol{n}_4}{6|K_L|}, & \boldsymbol{u}_\Lambda &= \frac{-1}{48|K_L|}\Big[(\boldsymbol{x}_{13}\times\boldsymbol{n}_4)\times\boldsymbol{x}_{L1}+(\boldsymbol{x}_{23}\times\boldsymbol{n}_4)\times\boldsymbol{x}_{L2}\Big], &&\text{on } K_L,\\
\boldsymbol{w}_\Lambda &= -\frac{(\boldsymbol{x}_{41}+\boldsymbol{x}_{42})\times \boldsymbol{n}_3}{6|K_R|}, & \boldsymbol{u}_\Lambda &= \frac{-1}{48|K_R|}\Big[(\boldsymbol{x}_{14}\times\boldsymbol{n}_3)\times\boldsymbol{x}_{R1}+(\boldsymbol{x}_{24}\times\boldsymbol{n}_3)\times\boldsymbol{x}_{R2}\Big], &&\text{on } K_R,\\
\boldsymbol{w}_\Lambda &= \Big[\frac{(\boldsymbol{x}_{31}+\boldsymbol{x}_{32})\times \boldsymbol{n}_4}{6|K|} & \boldsymbol{u}_\Lambda &= \frac{1}{48|K|}\Big[(\boldsymbol{x}_{13}\times\boldsymbol{n}_4)\times\boldsymbol{x}_{41}+(\boldsymbol{x}_{23}\times\boldsymbol{n}_4)\times\boldsymbol{x}_{42}\\
&\quad+\frac{(\boldsymbol{x}_{41}+\boldsymbol{x}_{42})\times \boldsymbol{n}_3}{6|K|}\Big], & &\quad+(\boldsymbol{x}_{14}\times\boldsymbol{n}_3)\times\boldsymbol{x}_{31}+(\boldsymbol{x}_{24}\times\boldsymbol{n}_3)\times\boldsymbol{x}_{32}\Big], &&\text{on } K.
\end{aligned}$$

Next, let us derive the liftings associated with the pressure. To treat this case, consider a face F (shared by the tetrahedra K_F^+ and K_F^- – see Figure 3). Let p_F denote the indicator function of edge F. The liftings $\mathbf{w}_F := \mathbf{w}(p_F)$ and $\mathbf{u}_F := \mathbf{u}(p_F)$ are supported on $K_F^+ \cup K_F^-$. Let $\boldsymbol{x}_i^\pm$, $i = 1,\dots,4$ denote any enumeration of the four vertices of $K_F^\pm$. In accordance with our previous notation, set $\boldsymbol{x}_{K_F^\pm}$ equal to the barycenter of $K_F^\pm$ and $\boldsymbol{x}_{iK} = \boldsymbol{x}_i^\pm - \boldsymbol{x}_{K_F^\pm}$. We can express the liftings on $K_F^\pm$ by

$$\mathbf{w}_F(\boldsymbol{x})|_{K_F^\pm} = \mathbf{w}^\pm \times (\boldsymbol{x}-\boldsymbol{x}_{K_F^\pm}), \qquad \mathbf{u}_F(\boldsymbol{x})|_{K_F^\pm} = \frac{1}{40}\sum_{i=1}^{4}\boldsymbol{x}_{iK}\times(\mathbf{w}^\pm\times\boldsymbol{x}_{iK}).$$

where

$$\mathbf{w}^\pm = -\frac{|F|}{2|K_F^\pm|}\boldsymbol{n}_F^\pm$$

and $\boldsymbol{n}_F^\pm$ denotes the outward unit normal of $K_F^\pm$ on F (see Figure 3).

The formulae for the maps associated with the body force are similar. If $\boldsymbol{f}$ is supported only on K, then $\mathbf{w}(\boldsymbol{f})$ and $\mathbf{u}(\boldsymbol{f})$ are supported only on K. Their values on K are given by

$$\mathbf{w}(\boldsymbol{f}) = \mathbf{w} \times (\boldsymbol{x} - \boldsymbol{x}_K), \qquad \mathbf{u}(\boldsymbol{f}) = \frac{1}{40} \sum_{i=1}^{4} \boldsymbol{x}_{iK} \times (\mathbf{w} \times \boldsymbol{x}_{iK}),$$

where

$$\mathbf{w} = \frac{1}{2|K|} \int_K \boldsymbol{f} \, \mathrm{d}x.$$

Now that we have expressions for the liftings of the basis functions, we can easily compute the local stiffness matrices of the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ with respect to the basis. Once the local matrices are made, one assembles them to get the global matrices in much the same way as one does for standard finite element methods. To compute the local stiffness matrix, we first list the degrees of freedom local to an element. In this list, we include the omitted elements of $\hat{\Lambda}_h$. The omissions can be taken care of after assembly by simply deleting the rows and columns corresponding to the omitted elements of $\hat{\Lambda}_h$. To geometrically identify the degrees of freedom on an element K, let $\boldsymbol{x}_i$ denote the vertices of K and E_{ij} denote the edge of K with endpoints $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$. There are six wedge degrees of freedom interior to K, which we denote by Λ_{ij} for $ij \in \mathcal{I}_0 := \{12, 13, 14, 23, 24, 34\}$. The wedge Λ_{ij} is geometrically identified as the wedge contained in ∂K with edge E_{ij}. In addition, there are twelve degrees of freedom from wedges "exterior" to K that contribute to the local stiffness matrix of K. We denote these as Λ_{ijk} for $ijk \in \mathcal{I}_1 := \{ijk : ij \in \mathcal{I}_0$ and k does not equal i or $j\}$ (cf. [7, Figure 4]). The wedge Λ_{ijk} is the (unique) wedge with edge E_{ij}, whose one face coincides with the face of K formed by vertices $\boldsymbol{x}_i, \boldsymbol{x}_j$ and $\boldsymbol{x}_k$, and whose other face is not contained in ∂K. Thus all wedge degrees of freedom within an element can be identified using the index set $\mathcal{I} = \mathcal{I}_0 \cup \mathcal{I}_1$. The pressure degrees of freedom are easier to enumerate: There is one for each face of K, so they can be identified using the index set $\mathcal{L} := \{1, 2, 3, 4\}$. The local stiffness matrices associated to an element K can now be given by

$$\begin{aligned}
\mathsf{A}^{(K)}_{IJ} &= \int_K \boldsymbol{w}(\boldsymbol{\psi}_{\Lambda_I}) \cdot \boldsymbol{w}(\boldsymbol{\psi}_{\Lambda_J}) \, \mathrm{d}x, && I, J \in \mathcal{I},\\
\mathsf{B}^{(K)}_{LJ} &= -\int_K \mathbf{curl}\, \mathbf{w}(p_L) \cdot \boldsymbol{u}(\boldsymbol{\psi}_{\Lambda_J}) \, \mathrm{d}x, && J \in \mathcal{I}, \quad L \in \mathcal{L},\\
\mathsf{C}^{(K)}_{LM} &= \int_K \mathbf{w}(p_L) \cdot \mathbf{w}(p_M) \, \mathrm{d}x, && L, M \in \mathcal{L}.
\end{aligned}$$

Here, as before, p_L denotes the characteristic function of the face F_L for all $L \in \mathcal{L}$.

We can calculate the integrals above after substituting the previously given expressions for the liftings of the basis functions in the integrands. To take into account modifications required near the boundary $\partial\Omega$, let σ_j equal zero if the face F_j is contained in the boundary $\partial\Omega$ and equal one otherwise. The simplified expressions for $\mathsf{A}^{(K)}, \mathsf{B}^{(K)}$ and $\mathsf{C}^{(K)}$ for any element K are given below. Suppose that $\{i, j, k, l\}$ is any permutation of $\{1, 2, 3, 4\}$. Then define

$$\boldsymbol{W}_I = \begin{cases} \sigma_l(\boldsymbol{x}_{ki} + \boldsymbol{x}_{kj}) \times \boldsymbol{n}_l + \sigma_k(\boldsymbol{x}_{li} + \boldsymbol{x}_{lj}) \times \boldsymbol{n}_k, & \text{if } I = ij, \\ -\sigma_l(\boldsymbol{x}_{ki} + \boldsymbol{x}_{kj}) \times \boldsymbol{n}_l, & \text{if } I = ijk, \end{cases}$$

$$\boldsymbol{U}_I = \begin{cases} \sigma_l(\boldsymbol{x}_{ik} \times \boldsymbol{n}_l) \times \boldsymbol{x}_{li} + (\boldsymbol{x}_{jk} \times \boldsymbol{n}_l) \times \boldsymbol{x}_{kj} \; + & \\ \qquad \sigma_k(\boldsymbol{x}_{il} \times \boldsymbol{n}_k) \times \boldsymbol{x}_{ki} + (\boldsymbol{x}_{jl} \times \boldsymbol{n}_k) \times \boldsymbol{x}_{lj}, & \text{if } I = ij, \\ -\sigma_l(\boldsymbol{x}_{ik} \times \boldsymbol{n}_l) \times \boldsymbol{x}_{li} + (\boldsymbol{x}_{jk} \times \boldsymbol{n}_l) \times \boldsymbol{x}_{kj}, & \text{if } I = ijk, \end{cases}$$

After a few simplifications, one finds that

$$\mathsf{A}^{(K)}_{IJ} = \frac{1}{36|K|} \boldsymbol{W}_I \cdot \boldsymbol{W}_J,$$

$$\mathsf{B}^{(K)}_{LJ} = \frac{1}{48|K|} \boldsymbol{E}_L \cdot \boldsymbol{U}_J,$$

$$\mathsf{C}^{(K)}_{LM} = \frac{1}{80|K|} \sum_{\ell=1}^{4} (\boldsymbol{E}_L \times \boldsymbol{x}_{\ell K}) \cdot (\boldsymbol{E}_M \times \boldsymbol{x}_{\ell K}),$$

where $\boldsymbol{E}_L = \boldsymbol{n}_L |F_L|$ for all $L \in \mathfrak{L}$. Using these local matrices, it is quite easy to implement the lowest order case of our method, even for general tetrahedral meshes. For the variable degree and higher order case, one would need to select a good basis for the polynomial spaces involved on one element and then perform the above steps within a computer implementation. Our calculations above, besides showing the essential simplicity of our discretization in the lowest order case, also clarifies the data structures one would need in implementing the method.

6. Extension to other boundary conditions

Although the previously considered Dirichlet boundary condition on velocity is the most commonly occurring boundary condition in the Stokes problem, other types of boundary conditions are also encountered in practice. One can have boundary conditions on the pressure of the form

$$p = s,$$

and boundary conditions on tangential vorticity of the form

$$\boldsymbol{n} \times \boldsymbol{\omega} = \boldsymbol{r}.$$

Here s and $\boldsymbol{r}$ are functions prescribed on parts of the boundary $\partial\Omega$. We now show how one may incorporate such boundary conditions into our hybridized discretization. Note that the above types of boundary conditions are difficult to impose in a natural fashion in many existing methods – see remarks in [11, § 4.3] and [12]. They are often practically important. E.g., pressure is often used as an outflow condition. The tangential vorticity boundary condition is useful when matching an exterior potential flow since vorticity is known to decay faster than velocity. The tangential vorticity boundary condition has been considered previously in [8] in formulations with the stream function.

Assume that the polyhedral boundary $\partial\Omega$ is partitioned into three disjoint subsets Γ_1, Γ_2, and Γ_3, such that each mesh face $F \in \mathfrak{F}$ on the boundary $\partial\Omega$ is contained in one and only one of these three subsets. We consider the Stokes equations (1)–(2) with the following boundary conditions:

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{g} && \text{on } \Gamma_1, \\ \left.\begin{aligned}\boldsymbol{n}\times\boldsymbol{\omega} &= \boldsymbol{r} \\ \boldsymbol{u}\cdot\boldsymbol{n} &= g_n\end{aligned}\right\} & && \text{on } \Gamma_2, \\ \left.\begin{aligned} p &= s \\ \boldsymbol{u}_{\mathsf{T}} &= \boldsymbol{g}_{\mathsf{T}}\end{aligned}\right\} & && \text{on } \Gamma_3. \end{aligned}$$

A straightforward generalization of our method can be obtained in this case.

To describe it, we first redefine the jump-functions as follows: The functions $[\![\boldsymbol{n}\cdot\boldsymbol{v}]\!]$ and $[\![\boldsymbol{n}\times\boldsymbol{\tau}]\!]$ are defined just as before on the interior faces, but for mesh faces F on the boundary, we set

$$[\![\boldsymbol{n}\cdot\boldsymbol{v}]\!]_F = \begin{cases} 0 & \text{for all faces } F \subseteq \Gamma_3, \\ \boldsymbol{n}\cdot\boldsymbol{v} & \text{for the remaining faces } F \subseteq \partial\Omega\setminus\Gamma_3, \end{cases}$$

and

$$[\![\boldsymbol{n}\times\boldsymbol{\tau}]\!]_F = \begin{cases} 0 & \text{for all faces } F \subseteq \Gamma_3\cup\Gamma_1, \\ \boldsymbol{n}\times\boldsymbol{\tau} & \text{for the remaining faces } F \subseteq \partial\Omega\setminus(\Gamma_3\cup\Gamma_1). \end{cases}$$

Then along the lines of the derivation of (13)–(16), we can derive the following hybridized mixed formulation: Find $(\boldsymbol{\omega}_h, \boldsymbol{u}_h, \boldsymbol{\lambda}_h, p_h) \in W_h \times V_h \times M_h \times P_h$ satisfying

$$\begin{aligned} (\boldsymbol{\omega}_h, \boldsymbol{\tau}_h)_\Omega - (\boldsymbol{u}_h, \mathbf{curl}\,\boldsymbol{\tau}_h)_\Omega - \sum_{F\in\mathcal{F}} (\boldsymbol{\lambda}_h, [\![\boldsymbol{n}\times\boldsymbol{\tau}_h]\!])_F &= (\boldsymbol{g}_{\mathsf{T}}, \boldsymbol{n}\times\boldsymbol{\tau}_h)_{\Gamma_1\cup\Gamma_3}, \\ (\boldsymbol{v}_h, \mathbf{curl}\,\boldsymbol{\omega}_h)_\Omega + \sum_{F\in\mathcal{F}} (p_h, [\![\boldsymbol{v}_h\cdot\boldsymbol{n}]\!])_F &= (\boldsymbol{f}, \boldsymbol{v}_h)_\Omega \\ &\quad - (s, \boldsymbol{v}_h\cdot\boldsymbol{n})_{\Gamma_3} \\ \sum_{F\in\mathcal{F}} (q_h, [\![\boldsymbol{u}_h\cdot\boldsymbol{n}]\!])_F &= (g_n, q_h)_{\Gamma_1\cup\Gamma_2}, \\ \sum_{F\in\mathcal{F}} (\boldsymbol{\mu}_h, [\![\boldsymbol{n}\times\boldsymbol{\omega}_h]\!])_F &= (\boldsymbol{\mu}_h, \boldsymbol{r})_{\Gamma_2}, \end{aligned}$$

for all $\boldsymbol{\tau}_h \in W_h, \boldsymbol{v}_h \in V_h, q_h \in P_h, \boldsymbol{\mu}_h \in M_h$. Here W_h and V_h are the same spaces as before. The spaces of Lagrange multipliers P_h and M_h continue to be defined by (11) and (12), but now with the revised definition of jump-functions.

For this formulation, we can prove, by a minor modification of the argument used in Proposition 2.1, that there is one and only one solution. Moreover, the entire analysis of Section 3 goes through with minor changes. We obtain a reduced Lagrange multiplier system and can formulate a theorem entirely analogous to Theorem 3.1. The discussion of the liftings and the basis functions in the previous sections continue to apply for these boundary conditions.

The method we presented in this paper gives a powerful alternative for problems in computational fluid mechanics which require exactly divergence free solutions for their successful treatment. Applications to such problems, the error analysis of the method, and the design of good preconditioners for solving the resulting matrix equations, are subjects of ongoing work.

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