Stability Analysis for Acoustic Waveguides: Impedance Boundary Conditions[1]

Leszek Demkowicz[a] Jay Gopalakrishan[b], and Norbert Heuer[c]

Dedicated to the memory of Professor Ivo Babuška

[a]Oden Institute, The University of Texas at Austin, USA
[b] Portland State University, Portland, OR 97207-0751, USA
[c] Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Santiago, Chile

Abstract

A model two-dimensional acoustic waveguide with lateral impedance boundary conditions (and outgoing boundary conditions at the waveguide outlet) is considered. The governing operator is proved to be bounded below with a stability constant inversely proportional to the length of the waveguide. The presence of impedance boundary conditions leads to a non self-adjoint operator which considerably complicates the analysis. The goal of this paper is to elucidate these complications, and tools that are applicable, as simply as possible. This work is a continuation of prior waveguide studies (where selfadjoint operators arose) by Melenk et al., "Stability Analysis for Electromagnetic Waveguides. Part 1: Acoustic and Homogeneous Electromagnetic Waveguides" (2023) [9], and Demkowicz et al. "Stability Analysis for Acoustic and Electromagnetic Waveguides. Part 2: Non-homogeneous Waveguides (2023) [4].

Key words: acoustic waveguides, well-posedness analysis

AMS classification: 78A50, 35Q61

1 Introduction

Typical acoustic waveguides have a length l which is many times the wavelength of the waves propagating within it. While solving a Helmholtz boundary value problem within the waveguide, it is of interest to know how the stability depends on l. Methods like the DPG method [2, 3] rely on a well-posedness estimate, or stability estimate, for the undiscretized problem. This paper is devoted to understanding this dependence for waveguides with impedance boundary conditions that give rise to non self-adjoint operators. It is the third part of a series of papers devoted to the stability analysis of acoustic and electromagnetic (EM) waveguides—see the first part [9] for further motivations driving this study.

A more specific motivation comes from the analysis of circular waveguides. Contrary to straight *open* waveguides where boundary conditions (BC) at infinity are replaced with a finite energy assumption, the analysis of *open circular waveguides* calls for the imposition of a radiation condition at $r = \infty$. The analysis of a circular waveguide illustrated in Fig.1, with an *impedance* BC at $r = b$, is the usual stepping stone towards the analysis of the open circular waveguide. The analysis

[1]To appear in *Appl. Mat.*, Institute of Mathematics, Czech Academy of Sciences

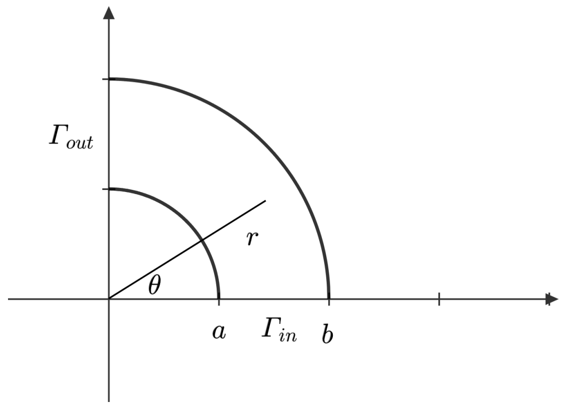

Figure 1: A circular acoustic waveguide

of a *straight* waveguide with an impedance BC presented in this paper is thus 'a stepping stone' for a more complicated 'stepping stone'. Separation of variables for a circular waveguide with an impedance BC leads to the Bessel equation:

$$r(rR')' + \omega^2 r^2 R = k^2 R,$$

with an arbitrary *complex order* $k \in \mathbb{C}$. It is much harder to analyze than the straight waveguide studied in this paper.

Formulation of the Problem

The problem of interest is illustrated in Fig. 2. The domain is $\Omega = I \times (0, l)$ where $I = (0, a)$. We are looking for pressure $p(x, z)$ and velocity field $u(x, z)$ satisfying the system of linear time-harmonic acoustic equations:

$$\begin{cases} i\omega p + \operatorname{div} u &= f \\ i\omega u + \boldsymbol{\nabla} p &= g\,. \end{cases} \tag{1.1a}$$

The system is accompanied with the following Boundary Conditions (BC) where u_n denotes the exterior normal component of u:

- hard BC on the left-hand side of the lateral boundary:

$$u_n = 0 \quad \text{on } \Gamma_u := \{(0, z) \,:\, z \in (0, l)\}\,, \tag{1.1b}$$

- impedance BC on the right-hand side of the lateral boundary:

$$u_n = -dp \quad \text{on } \Gamma_{imp} := \{(a, z) \,:\, z \in (0, l)\}\,, \tag{1.1c}$$

- soft BC on the inflow boundary:

$$p = 0 \quad \text{on } \Gamma_{in} := \{(x, 0) \,:\, x \in (0, a)\}\,, \tag{1.1d}$$

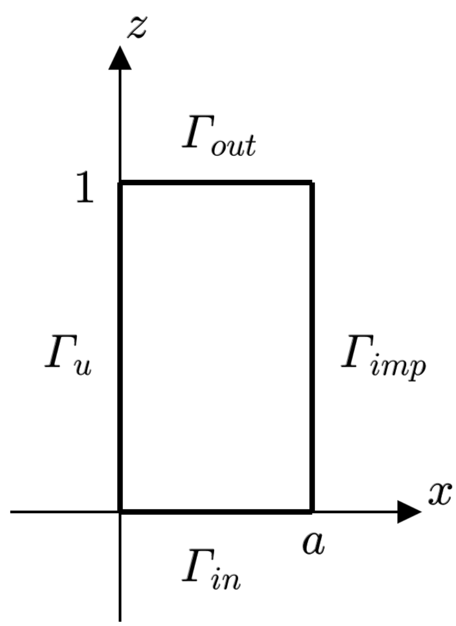

Figure 2: Acoustic waveguide problem.

- a non-local Dirichlet-to-Neumann (DtN) BC on the outflow boundary:

$$u_n = \text{DtN}p \quad \text{on } \Gamma_{out} := \{(x,l) : x \in (0,a)\}\,. \tag{1.1e}$$

  The definition of the DtN BC involves the decomposition of the solution into modes (to be introduced), and is explained below.

Here, $\omega > 0$ is the angular frequency (we are using the $e^{-i\omega t}$ ansatz in time), and $d > 0$ is an impedance constant. The DtN condition secures that the wave is outgoing. Formulation of the DtN condition involves the use of propagation modes and in essence forces the analysis of the problem using the modal decomposition. The goal of this paper is to show that the operator governing the equations is bounded below, namely

$$\|p\|^2_{L^2(\Omega)} + \|u\|^2_{(L^2(\Omega))^2} \leq C(\|f\|^2_{L^2(\Omega)} + \|g\|^2_{(L^2(\Omega))^2})\,, \tag{1.2}$$

and to investigate the dependence of the constant C upon the waveguide length l. Note that C may depend on ω, a dependence we do not explicitly track.

Reduction to the Helmholtz problem. Let $q \in H^1(\Omega)$, $q = 0$ on Γ_{in}, be a test function. We multiply (1.1a)$_1$ with factor $i\omega$, then with test function $\bar{q}$, integrate over the domain Ω, integrate by parts, and incorporate BCs to obtain

$$-\omega^2(p,q) - i\omega(u, \boldsymbol{\nabla} q) - i\omega d\langle p,q\rangle_{\Gamma_{imp}} + i\omega\langle \text{DtN}p, q\rangle_{\Gamma_{out}} = i\omega(f,q)\,.$$

Next we multiply (1.1a)$_2$ with the gradient of the test function $\boldsymbol{\nabla}\bar{q}$ and integrate over the domain Ω to obtain

$$i\omega(u, \boldsymbol{\nabla} q) + (\boldsymbol{\nabla} p, \boldsymbol{\nabla} q) = (g, \boldsymbol{\nabla} q)\,.$$

Summing up the two equations, we obtain the Helmholtz problem in its variational form,

$$
\left\{
\begin{array}{l}
p \in H^1(\Omega) : \ p = 0 \text{ on } \Gamma_{in}, \\
(\boldsymbol{\nabla} p, \boldsymbol{\nabla} q) - \omega^2 (p, q) - i\omega d \langle p, q \rangle_{\Gamma_{imp}} + i\omega \langle \mathrm{DtN} p, q \rangle_{\Gamma_{out}} = i\omega (f, q) + (g, \boldsymbol{\nabla} q) \\
\hfill q \in H^1(\Omega), \ q = 0 \text{ on } \Gamma_{in} .
\end{array}
\right.
$$

The right-hand side above represents a linear and continuous functional on $H^1(\Omega)$ and, for convenience, we will replace it with its Riesz representation $r \in H^1(\Omega)$,

$$
\left\{
\begin{array}{l}
p \in H^1(\Omega) : \ p = 0 \text{ on } \Gamma_{in}, \\
(\boldsymbol{\nabla} p, \boldsymbol{\nabla} q) - \omega^2 (p, q) - i\omega d \langle p, q \rangle_{\Gamma_{imp}} + i\omega \langle \mathrm{DtN} p, q \rangle_{\Gamma_{out}} = (r, q)_{H^1(\Omega)} \\
\hfill q \in H^1(\Omega), \ q = 0 \text{ on } \Gamma_{in} .
\end{array}
\right.
\tag{1.3}
$$

Note that

$$
\|r\|_{H^1(\Omega)} \le (\omega^2 \|f\|^2_{L^2(\Omega)} + \|g\|^2_{(L^2(\Omega))^2})^{\frac{1}{2}} .
$$

It is easy to show that the original boundedness below condition (1.2) is equivalent to

$$
\|p\|_{H^1(\Omega)} \le C \|r\|_{H^1(\Omega)} . \tag{1.4}
$$

The modes. We seek first solutions to the homogeneous system satisfying only the hard and impedance BCs. Assuming an exponential ansatz in z, we look for the solution in the form

$$
p = p(x) e^{i\beta z} \quad \text{and} \quad u = (u_x, u_z) \quad \text{with} \quad u_x = u_x(x) e^{i\beta z}, \ u_z = u_z(x) e^{i\beta z} .
$$

Above, we have overloaded symbols for p, u_x, u_z. It should be clear from the context which functions we have in mind. Substituting the ansatz into the equations, we obtain a system of three first order ordinary differential equations (ODEs) for unknowns $p(x), u_x(x), u_z(x)$,

$$
\left\{
\begin{array}{rll}
i\omega p + u_x' + i\beta u_z & = 0 & (i\omega \bar{q}, \int, \text{relax}) \\
i\omega u_x + p' & = 0 & (\bar{q}', \int) \\
i\omega u_z + i\beta p & = 0 & (-i\beta \bar{q}, \int) .
\end{array}
\right.
\tag{1.5}
$$

The system is accompanied with BCs:

- hard BC on Γ_u: $u_x = 0$,
- impedance BC on Γ_{imp}: $u_x = -dp$.

Multiplying the equations with terms indicated in (1.5), relaxing the first equation[2], and adding the equations, we obtain a variational eigenvalue problem for a mode p (recall that $I = (0, a)$):

$$
\left\{
\begin{array}{l}
\text{Find } p \in H^1(I) \setminus \{0\}, \ \beta^2 \in \mathbb{C} \text{ such that} \\
\displaystyle \int_0^a \{p' \bar{q}' + p \bar{q}\} \, dx - i\omega d \, p(a) \overline{q(a)} = (\underbrace{\omega^2 - \beta^2 + 1}_{=\lambda}) \int_0^a p \bar{q} \, dx \qquad \forall q \in H^1(I) .
\end{array}
\right.
\tag{1.6}
$$

[2]By relaxing we mean integrating by parts and incorporating the BCs.

The propagation constant β is related to the (complex) eigenvalue λ by $\beta^2 = \omega^2 + 1 - \lambda$. For impedance constant $d = 0$, the sesquilinear form on the left corresponds to the 1D Laplacian with Neumann BCs which possesses a sequence of real, non-negative eigenvalues $\lambda_n \to \infty$. We have thus a finite number of propagating modes ($\beta^2 > 0$) followed by an infinite number of evanescent modes ($\beta^2 \leq 0$). The eigenvalue problem can also be formulated using the language of closed operators. Introducing the operator,

$$A : L^2(I) \supset D(A) \ni p \mapsto Ap = -p'' + p \in L^2(I) \tag{1.7a}$$

with

$$\begin{aligned} D(A) :=& \{p \in L^2(I) : \; p'' \in L^2(I), \, p'(0) = 0, \; p'(a) = i\omega d p(a)\} \\ =& \{p \in H^2(I) : \; p'(0) = 0, \; p'(a) = i\omega d p(a)\} \end{aligned} \tag{1.7b}$$

we can restate the eigenvalue problem as

$$Ap = \lambda p \, . \tag{1.7c}$$

The operator is formally self-adjoint but it is not self-adjoint if $d \neq 0$, as in that case

$$D(A^*) = \{p \in H^2(I) : \; p'(0) = 0, \; p'(a) = -i\omega d p(a)\} \neq D(A) \, .$$

The non-local DtN boundary operator. The same modes are needed to formulate the *Dirichlet-to-Neumann* operator present in the BC at Γ_{out}. Extending the waveguide all the way to infinity, we seek the solution of the *homogeneous* waveguide problem for $z > l$ with an outgoing (or radiation) BC at infinity. The solution is of the form:

$$p = \sum_j p_j X_j(x) e^{i\beta_j z}, \quad p_j \in \mathbb{C} \, .$$

Note that the contributions $e^{-i\beta_j z}$ have been eliminated by the outgoing radiation condition at $z = \infty$. The DtN operator is now readily obtained:

$$\text{DtN} : \sum_j p_j X_j(x) e^{i\beta_j z} = p \mapsto \frac{\partial p}{\partial z} = \sum_j i\beta_j p_j X_j(x) e^{i\beta_j z} \, .$$

For a single mode X_j, the DtN BC reduces thus to an impedance BC with a mode dependent impedance constant $i\beta_j$. In order to represent the BC in terms of velocity, we expand first the z-component of the velocity in the same nodes,

$$u_z = \sum_j u_{z,j} X_j(x) e^{i\beta_j z},$$

and extend equation $(1.1a)_2$ to the boundary to obtain the relation relating the spectral components of the velocity and pressure:

$$-i\omega u_{z,j} = i\beta_j p_j \, .$$

See [9] for a detailed mathematical discussion of the DtN operator for the self-adjoint case.

The strategy. Let us assume for a moment that the eigenvalue problem (1.6) admits a sequence of eigenvectors X_n with corresponding eigenvalues λ_n and β_n^2. The conjugate $\bar{X}_n$ represents the eigenvectors of the adjoint A^*. Let us also assume for simplicity that all eigenvalues are simple and distinct[3]. We seek the solution to the Helmholtz problem (1.3) in the form

$$p(x,z) = \sum_j X_j(x) p_j(z)\,.$$

Substituting the ansatz into the variational equation (1.3) and testing with $\overline{\bar{X}_k q(z)} = X_k \overline{q(z)}$ we obtain a system of decoupled 1D variational Helmholtz problems for the spectral components $p_k(z)$:

$$\begin{cases} p_k \in H^1_{(0}(0,l) \\ \displaystyle\int_0^l \{p_k' \bar{q}' - \beta_k^2 p_k \bar{q}\}\, dz + i\beta_k p_k(l)\bar{q}(l) = \int_0^l (r, \bar{X}_k)_{H^1(I)} \bar{q}\, dz + \int_0^l (\frac{\partial r}{\partial z}, \bar{X}_k)_{L^2(I)} \bar{q}'\, dz, \\ \hfill q \in H^1_{(0}(0,l) \end{cases}$$

where

$$H^1_{(0}(0,l) := \{q \in H^1(0,l) :\ q(0) = 0\}\,.$$

The following stability estimate can be found in [9] (Lemma 4),

$$\int_0^l |p_k'|^2\, dz + |\beta_k|^2 \int_0^l |p_k|^2\, dz \lesssim l^2 \left\{ \int_0^l |(\frac{\partial r}{\partial z}, \bar{X}_k)_{L^2(I)}|^2 + \frac{1}{|\beta_k|^2} \int_0^l |(r, \bar{X}_k)_{H^1(I)}|^2 \right\} dz. \tag{1.8}$$

What makes the estimate non-trivial, is the presence of β_k factors on both left- and right-hand sides of the inequality, and the explicit dependence of the stability constant upon the length l. Here and throughout, we say that $A \lesssim B$ for two quantities A and B when there is a constant $C > 0$ *independent of* l such that the inequality $A \leq CA$ holds.

Estimation for the self-adjoint case. For vanishing impedance, $d = 0$, operator A is self-adjoint, and the corresponding eigenvectors are simultaneously orthogonal and complete in $L^2(I)$

[3] This is indeed the case for our model problem.

and $H^1(I)$. Normalizing them in the L^2-norm, we proceed as follows [9]:

$$
\begin{aligned}
&\| \sum_j p_j X_j \|^2_{H^1(\Omega)} \\
&= \int_0^l \left\{ \| \sum_j p_j X_j \|^2_{H^1(I)} + \| \sum_j p_j' X_j \|^2_{L^2(I)} \right\} dz && \text{(definition of } H^1(\Omega)\text{-norm)} \\
&= \sum_j \left\{ \int_0^l \lambda_j |p_j(z)|^2 \, dz + \int_0^l |p_j'(z)|^2 \, dz \right\} && \begin{pmatrix} H^1(I) \text{ and } L^2(I)- \\ \text{orthogonality of } X_j \\ \|X_j\|^2_{L^2(I} = 1 \\ \|X_j\|^2_{H^1(I)} = \lambda_j \end{pmatrix} \\
&\lesssim l^2 \sum_j \left\{ \int_0^l |(\frac{\partial r}{\partial z}, X_j)_{L^2(I)}|^2 \, dz + \frac{1}{\lambda_j} \int_0^l |(r, X_j)_{H^1(I)}|^2 \, dz \right\} && \begin{pmatrix} \text{1D Helmholtz stability} \\ \beta_j \approx \lambda_j \end{pmatrix} \\
&= l^2 \sum_j \left\{ \int_0^l |(\frac{\partial r}{\partial z}, X_j)_{L^2(I)}|^2 \, dz + \int_0^l |(r, \frac{X_j}{\|X_j\|_{H^1(I)}})_{H^1(I)}|^2 \, dz \right\} && (\|X_j\|^2_{H^1(I)} = \lambda_j) \\
&= l^2 \int_0^l \left\{ \sum_j |(\frac{\partial r}{\partial z}, X_j)_{L^2(I)}|^2 + \sum_j |(r, \frac{X_j}{\|X_j\|_{H^1(I)}})_{H^1(I)}|^2 \right\} dz \\
&= l^2 \int_0^l \left\{ \|\frac{\partial r}{\partial z}\|^2_{L^2(I)} + \|r\|^2_{H^1(I)} \right\} && \text{(orthogonality)} \\
&= l^2 \|r\|^2_{H^1(\Omega)} \, .
\end{aligned}
$$

The goal of this paper is to extend this analysis to the non self-adjoint case for $d > 0$.

Scope of the paper. In Section 2, we review several fundamental results from the theory of non self-adjoint operators relevant to our problem. Section 3 is devoted to the analysis of the 1D eigenvalue problem with the impedance BC. Although a similar such 1D problem was studied by J. Schwartz 70 years ago [10], we include, for completeness, some details of the 1D results needed for analysis of the 2D waveguide. In particular, we analyze eigenbasis expansions both in H^1 and L^2. We also show how to leverage the 1D estimates to apply the Glazman theorem (which seems to not have been available at the time of Schwartz's writing). The final 2D stability estimate, tracking dependence on waveguide length, is presented in Section 4, which to the best of our knowledge, is original. The paper concludes with a short summary of our results in Section 5.

2 Fundamental Results on Non-selfadjoint Operators

In this section, we recall a few fundamental concepts (see e.g., [6]) that will be useful for analyzing our non-selfadjoint waveguide problem. Let X be a separable Banach space. A sequence $\phi_j \in X, j = 1, 2, \ldots$, is a *Schauder basis* for space X if

$$\forall\, x \in X \quad \exists!\, x_j \in \mathbb{C},\, j = 1, 2, \ldots \quad : \quad x = \sum_{j=1}^{\infty} x_j \phi_j \, .$$

Thus, given a Schauder basis, the coefficients in the basis expansion, i.e., the numbers x_j above, exist and are unique. Moreover the partial sums of the above infinite sum converges in the norm of the Banach space. The simplest example of a Schauder basis for a Hilbert space is an orthonormal basis. The basic properties of Schauder bases are summarized next (see, e.g., [6, p. 306]).

THEOREM 1

[(Schauder,Banach)]

Let $(\phi_j)_j$ be a Schauder basis for a Hilbert space X. The following holds.

- *There exists a biorthogonal sequence $(\psi_j)_j$, i.e., $(\phi_j, \psi_k)_X = \delta_{jk}$.*
- *"Linear independence" of vectors ϕ_j: $\phi_j \notin \overline{\mathrm{span}\{\phi_k \, : \, k \neq j\}}$.*
- *Sequence $(\phi_j)_j$ is* complete *in X, i.e., $\overline{\mathrm{span}\{\phi_j\}} = X$.*
- *Sequence $(\psi_j)_j$ is also a Schauder basis for space X.*

∎

In the remainder, unless otherwise stated, X denotes a Hilbert space.

Riesz basis. A sequence $\phi_j \in X, j = 1, 2, \ldots$, is a *Riesz basis* for a Hilbert space X if there exists a linear bounded operator $A : X \to X$ with a bounded inverse such that

$$\phi_j = A\chi_j$$

for some orthonormal basis $\chi_j, j = 1, 2, \ldots$.

THEOREM 2

[Bari]

The following conditions are equivalent to each other.

(i) Sequence $(\phi_j)_j$ is a Riesz basis.

(ii) Sequence $(\phi_j)_j$ represents an orthonormal basis in an inner product norm equivalent to the original inner product in X.

(iii) Sequence $(\phi_j)_j$ is complete in X, and there exist positive constants α_1, α_2 such that

$$\alpha_1 \sum_{j=1}^{n} |x_j|^2 \leq \| \sum_{j=1}^{n} x_j \phi_j \|^2 \leq \alpha_2 \sum_{j=1}^{n} |x_j|^2 \tag{2.1}$$

for any $n > 0$, and any sequence of complex numbers x_j, $j = 1, \ldots, n$.

∎

Proof: See [1] or [6, p. 310]. ∎

Dissipative operators. A linear operator: $X \supset D(A) \ni x \mapsto Ax \in X$ is called *dissipative* if

$$\Im(Ax, x) \geq 0, \quad x \in D(A)\,.$$

If A is bounded (and, therefore, defined on the whole X), then

$$\Im(Ax, x) = \frac{1}{2i}[(Ax, x) - \overline{(Ax, x)}] = (\frac{1}{2i}(A - A^*)x, x)\,,$$

so the condition is equivalent to the semi-positive definiteness of $\frac{1}{2i}(A - A^*)$.

THEOREM 3

[Glazman]

Let ψ_j, $j = 1, 2, \ldots$, be a system of unit eigenvectors corresponding to distinct eigenvalues λ_j of a dissipative operator such that

$$\sum_{\substack{j,\,k\,=\,1 \\ j\,\neq\,k}}^{\infty} \frac{\Im\lambda_j\,\Im\lambda_k}{|\lambda_j - \overline{\lambda_k}|^2} < \infty\,. \tag{2.2}$$

Then the system $(\psi_j)_j$ forms a Riesz basis for the closure of its span,

$$\overline{span\{\psi_j,\, j = 1, 2, \ldots\}}\,.$$

∎

Proof: See [5] or [6, p. 328]. ∎

Schatten class operators. Let $p \in [1, \infty)$. A compact operator $A : X \to X$ is in the p Schatten class, denoted by $\mathcal{C}_p$, if

$$\sum_{j=1}^{\infty} s_j^p < \infty$$

where s_j are *singular values* of operator A. One can show that the Schatten operators form a scale, i.e., $\mathcal{C}_p \subset \mathcal{C}_q$ for $p < q$. Operators in $\mathcal{C}_1$ are called nuclear operators, and those in $\mathcal{C}_2$ are the Hilbert–Schmidt operators. Defining

$$p(A) := \inf\{p \,:\, \sum_{j=1}^{\infty} |s_j|^p < \infty\},$$

we call A a *Schatten operator* if $p(A) < \infty$.

Consider the eigenvalue problem to find $\lambda_0 \in \mathbb{C}$ and $0 \neq x_0 \in X$ satisfying

$$(I - T)x_0 = \lambda_0 H x_0 \tag{2.3}$$

where $T : X \to X$ is an arbitrary compact operator, and $H : X \to X$ is an injective, compact, selfadjoint Schatten operator. The *associated eigenvectors* of such a λ_0 are $x_1, x_2, \ldots, x_m \in X$ satisfying

$$(I - T - \lambda_0 H)x_j = Hx_{j-1}, \quad j = 1, 2, \ldots, m, \tag{2.4}$$

and the *generalized eigenspace* of the operator pencil $L(\lambda) = I - T - \lambda H$, associated to such a λ_0, is the span of all such x_j, $j = 0, 1, 2, \ldots, x_m$. Then, the algebraic multiplicity of λ_0 is $m + 1$. The following result is a corollary of the well-known Keldyš theorems: see [8] or [6, p. 257–260] (cf. [7, Theorem 2.1]).

THEOREM 4

[Keldyš]

In the above setting, the sum of all the generalized eigenspaces of $L(\lambda)$ is dense in X. The spectrum of $L(\lambda)$ consists of an infinite sequence of eigenvalues, each of finite algebraic multiplicity, which do not accumulate in $\mathbb{C}$. If in addition H is non-negative, then, for any $\varepsilon > 0$, only finitely many eigenvalues lie outside the sector $\{z \in \mathbb{C} : |\arg z| < \varepsilon\}$. ∎

3 Analysis of the Eigenvalue Problem

In operator form eigenvalue problem (1.6) reads

$$\begin{cases} p \in H^1(I) \setminus \{0\},\ \lambda \in \mathbb{C} \\ Rp - Dp = \lambda M p \end{cases} \tag{3.1}$$

where $R : H^1(I) \to (H^1(I))'$ is the Riesz operator, $D : H^1(I) \to (H^1(I))'$ is an operator with a non-negative imaginary part and of finite rank, and M is a compact operator representing the composition of two embeddings $H^1(I) \hookrightarrow L^2(I)$ and $L^2(I) \hookrightarrow (H^1(I))'$, i.e.,

$$\langle Rp, q\rangle = (p, q)_{H^1(I)},$$
$$\langle Dp, q\rangle = i\omega d\, p(a)\overline{q}(a),$$
$$\langle Mp, q\rangle = (p, q)_{L^2(I)}.$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing in $H^1(I)$. Applying R^{-1} to both sides of (3.1), we get

$$\begin{cases} p \in H^1(I) \setminus \{0\},\ \lambda \in \mathbb{C} \\ (I - R^{-1}D)p = \lambda R^{-1} M p\,. \end{cases} \tag{3.2}$$

Putting $T = R^{-1}D$ and $H = R^{-1}M$, this fits into the setting of (2.3). Indeed, since D has finite rank, it is compact, and by the Rellich embedding, H is compact. Moreover, H is selfadjoint in $H^1(I)$ as

$$\begin{aligned}(Hp, q)_{H^1(I)} = (R^{-1}Mp, q)_{H^1(I)} &= \langle Mp, q\rangle = (p, q)_{L^2(I)} = \overline{(q, p)}_{L^2(I)} \\ &= \overline{\langle Mq, p\rangle} = \overline{(R^{-1}Mq, p)}_{H^1(I)} = (p, Hq)_{H^1(I)}.\end{aligned}$$

This also shows that H is positive definite (and hence injective). Finally, H is also a Schatten operator—in fact, it is easy to see that it is a Hilbert-Schmidt operator.

Hence, by Theorem 4, it follows that the spectrum of (3.2) consists of an infinite sequence of eigenvalues which do not accumulate in $\mathbb{C}$. Furthermore, every eigenvalue has finite algebraic multiplicity and, for any $\delta > 0$, there are only finitely many eigenvalues outside of the sector $\{z \in \mathbb{C} : |\arg(z)| < \delta\}$. We continue to refine this conclusion further.

Lemma 1
The algebraic multiplicity of every eigenvalue of (3.2) *is one.* ▮

Proof: Suppose $p \in H^1(I)$ solve (3.2) for some eigenvalue λ. Then

$$p'' + (\lambda - 1)p = 0, \qquad p'(0) = 0, \qquad p'(a) - i\omega d p(a) = 0. \tag{3.3}$$

By (2.4), an associated eigenvector $p_1 \in H^1(I)$, if it exists, satisfies $(I - T - \lambda H)p_1 = Hp$. Applying operator R to both sides,

$$(R - D - \lambda M)p_1 = Mp,$$

or, equivalently, for all $q \in H^1(I)$,

$$(p_1', q') + (p_1, q) - i\omega d p_1(a)\overline{q(a)} - \lambda(p_1, q) = (p, q).$$

For smooth enough q, after integration by parts, this implies

$$-\int_0^a p_1 \overline{(q'' + (\lambda - 1)q)} + p_1(a)\overline{(q'(a) + i\omega d q(a))} - p_1(0)\overline{q'(0)} = \int_0^a p\overline{q},$$

a relation which can be extended by density to eigenfunctions q. Substituting $q = p$ and using every equation of (3.3) after conjugating, we find that the left hand side vanishes, while the right hand side equals $\|p\|^2_{L^2(I))} \neq 0$. Hence p_1 cannot exist. ▮

Now, returning to the specific form of (3.2) given by the boundary value problem (3.3), we impose the boundary conditions on the general form of the solution $p(x) = c_1 \sin(\sqrt{\lambda - 1}x) + c_2 \cos(\sqrt{\lambda - 1}x)$. We conclude that $c_1 = 0$, $c_2 \neq 0$, and $z := a\sqrt{\lambda - 1}$ solves the following transcendental equation (similar to an equation studied by [10, §6]),

$$iz \tan z = \omega d a. \tag{3.4}$$

Note that if z is a root, then $-z$ is also a root of (3.4). We use the properties of z with positive real part to deduce the properties of the eigenvalues

$$\lambda - 1 = \frac{z^2}{a^2} = \omega^2 - \beta^2. \tag{3.5}$$

In the selfadjoint case of $d = 0$, the eigenvalues can easily be computed and enumerated as $z_n = n\pi$. The next lemma shows how z_n values are perturbed off the real axis in the $d > 0$ case.

Lemma 2
The roots of (3.4) *are simple, and the ones with positive real part, except possibly for a finite number*

of them, form a sequence $z_n = x_n + iy_n$*, as* $n \to \infty$*, with*

$$\begin{aligned} x_n &= n\pi + O(n^{-3}) \\ y_n &= -\frac{\omega da}{n\pi} + o(n^{-1}). \end{aligned} \tag{3.6}$$

∎

Proof: Fix an arbitrarily small $\delta > 0$. As we have seen, by Theorem 4, almost all the eigenvalues λ lie in the sector $|\arg(z)| < \delta$, and form a sequence λ_n with $|\lambda_n| \to \infty$ (since they cannot have an accumulation point). It follows that the roots $z = a\sqrt{\lambda - 1}$ of (3.4) with positive real part form a sequence

$$z_n = x_n + iy_n, \qquad \text{with } x_n \to \infty, \tag{3.7}$$

as $n \to \infty$. From relation (3.4) it is clear that, for sufficiently large integer n, there is at least one root in every strip $(n - 1/2)\pi < \Re z < (n + 1/2)\pi$ (since tan maps these strips onto $\mathbb{C} \setminus \{\pm i\}$ and $-i\omega da/z \neq \pm i$ is bounded there). Asymptotic uniqueness of roots within these strips will be shown below.

Equations for real and imaginary parts of z*.* Dropping the subscript n temporarily, let $z = x + iy$ be a root of (3.4). Taking the imaginary part of both sides of the variational formulation (1.6) after setting $q = p$ there, we conclude that $\Im\lambda < 0$. By (3.5), this gives

$$xy < 0, \qquad \text{so } y < 0 \text{ as } x \to +\infty \tag{3.8}$$

per (3.7). Calculating the real and imaginary parts of $\tan z$,

$$iz\tan z = i(x + iy)\frac{\sin 2x + i\sinh 2y}{\cos 2x + \cosh 2y} = i\frac{(x\sin 2x - y\sinh 2y) + i(x\sinh 2y + y\sin 2x)}{\cos 2x + \cosh 2y}. \tag{3.9}$$

Comparing with the real and imaginary parts of (3.4), we see that z solves (3.4) if and only if

$$x\sin 2x = y\sinh 2y, \tag{3.10a}$$

$$\text{and} \qquad x\sinh 2y + y\sin 2x = -\omega da(\cos 2x + \cosh 2y). \tag{3.10b}$$

Multiplicity of z*.* If z is a multiple root of $iz\tan z - \omega da$ then, by (3.10a), (x, y) is a multiple root of $f(x, y) = x\sin 2x - y\sinh 2y$. But $f(x, \cdot)$ is strictly concave with the only double root at 0, and $y \neq 0$. Therefore, the roots of (3.4) are simple.

Characterization of z *when* $x > \omega da$*.* Relation (3.10b) with y replaced through (3.10a) reads

$$x(\sinh^2 2y + \sin^2 2x) = -\omega da\sinh(2y)(\cos 2x + \cosh 2y). \tag{3.11}$$

By (3.8), we have $\sinh(2y) < 0$ as $x \to \infty$, so (3.11) implies $x\sinh^2 2y < -\omega da\sinh(2y)(1 + \cosh 2y)$. Hence

$$x < \omega da\frac{1 + \cosh 2y}{\sinh|2y|} = \omega da\frac{2 + v + v^{-1}}{v - v^{-1}} = \omega da\frac{2v + v^2 + 1}{v^2 - 1} = \omega da\frac{v + 1}{v - 1},$$

with $v = e^{2|y|}$. Equality holds iff $x > \omega da$ and $v = (x + \omega da)/(x - \omega da)$. If $x > \omega da$, then $1 < v = e^{2|y|} < (x + \omega da)/(x - \omega da)$ and

$$|y| = \frac{1}{2}\log v \le \frac{1}{2}\log\frac{x+\omega da}{x-\omega da}, \quad \text{so} \quad 0 < -y \to 0 \quad \text{as } x \to \infty. \tag{3.12}$$

Asymptotic behavior. Comparing signs in (3.10a) shows that x has a representation

$$x = n\pi + \delta x \quad \text{with } n \in \mathbb{N}_0,\ \delta x \in (0, \pi/2).$$

Setting $x = n\pi + \delta x$ in (3.10a), we obtain

$$\sin 2\delta x = \frac{y \sinh 2y}{x}.$$

Therefore, by (3.12), $\delta x \in (0, \pi/2)$ tends to 0 or $\pi/2$ when $x \to \infty$. If $\delta x \to \pi/2$, then $|z \tan z| \to \infty$ contradicting relation $iz \tan z = \omega da$. Therefore, $\delta x \to 0$ and

$$\frac{x\delta x}{y^2} \to 1 \quad \text{as} \quad x \to \infty.$$

Taking the limit $x \to \infty$ (implying $\delta x \to 0$ and $y \to 0$) in (3.11) reveals that

$$\lim_{x\to\infty}(xy + \delta x y) = \lim_{x\to\infty} xy = -\omega da. \tag{3.13}$$

Combination of the two asymptotics yields

$$\lim_{x\to\infty} x^3 \delta x = (\omega da)^2. \tag{3.14}$$

Relations (3.13), (3.14) can be resumed as

$$y = -\frac{\omega da}{x} + o(x^{-1}), \quad \delta x = \frac{(\omega da)^2}{x^3} + o(x^{-3}) \quad (x \to \infty).$$

Asymptotic uniqueness. We have seen that, asymptotically, $z = n\pi + \delta x + iy$ with $\delta x, y \to 0$ $(n \to \infty)$. To finish the proof of the lemma it remains to show that, given a sufficiently large n, the perturbations δx and y are unique. This follows from system (3.10). Given $n \in \mathbb{N}$, its solution $(\delta x, y)$ is the root of a function $F = (F_1, F_2)$ with derivative of order $2n\pi\,\mathrm{id}$ where $\mathrm{id} \in \mathbb{R}^{2\times 2}$ is the identity matrix. We select $n \ge n_0$ large enough so that $F'[\xi_1, \xi_2] := \begin{pmatrix} \operatorname{grad} F_1(\xi_1)^\top \\ \operatorname{grad} F_2(\xi_2)^\top \end{pmatrix}$ is invertible for $\xi_1, \xi_2 \in [0,1] \times [-1, 0]$ (to fix a compact set that contains $(\delta x, y)$ for $n \ge n_0$). If there are two roots $w_1,\ w_2 \in (0,1) \times (-1,0)$ for a strip with $n \ge n_0$, then the component-wise application of the mean value theorem implies the existence of $\xi_1,\ \xi_2$ on the line connecting w_1 and w_2 with $F'[\xi_1, \xi_2](w_2 - w_1) = 0$, that is, $w_1 = w_2$. ∎

We conclude with properties of the eigenvalue problem (3.2).

PROPOSITION 1

The operator $-R + i\omega dD$ from (3.2) (with opposite sign) is dissipative. Its eigenvalues are simple *and form a sequence (λ_n) with $|\lambda_n| \to \infty$ $(n \to \infty)$ satisfying the Glazman criterion*

$$\sum_{j\neq k} \frac{\Im\lambda_j \Im\lambda_k}{|\lambda_j - \overline{\lambda_k}|^2} < \infty.$$

∎

Proof: The variational form (1.6) of (3.2) gives $-\Im(\langle(R - i\omega dD)p, p\rangle_{(H^1(I))' \times H^1(I)} = \omega d|p(a)|^2 \geq 0$, showing dissipativity. We have seen that the eigenvalues λ of (3.2) are related to the roots z of (3.4) by $\lambda = \frac{z^2}{a^2} + 1$. By Lemma 2 the roots are simple and so are the eigenvalues λ. The eigenvalues form a sequence (λ_n) with $\lambda_n = a^{-2}(x_n^2 - y_n^2 + 2ix_n y_n) + 1$ where $x_n^2 - y_n^2 = n^2\pi^2 + O(n^{-2})$ and $2x_n y_n = -\omega da + o(1)$, $n \to \infty$. The Glazman criterion holds since

$$\sum_{j\neq k} \frac{\Im\lambda_j \Im\lambda_k}{|\lambda_j - \bar{\lambda}_k|^2} \leq C \sum_{j\neq k} \frac{1}{|j^2 - k^2|^2} < \infty$$

for a positive constant C. ∎

The fundamental results on nonselfadjoint operators from Section 2 lead to the following properties of the eigenfunctions of (3.2).

PROPOSITION 2

Eigenvalue problem (3.2) *possesses a sequence of eigenpairs* $(\lambda_j, X_j), j = 1, \ldots$. *The space generated by the eigenfunctions is dense in* $L^2(\Omega)$ *and* $H^1(\Omega)$. *If, in addition, the eigenfunctions are normalized in the* L^2*-norm, then there are constants* $c_1, c_2 > 0$ *such that for any sequence of complex coefficients* $u_j \in \mathbb{C}$, $j = 1, \ldots, N$, *and any* N,

$$\begin{aligned} c_1 \sum_{j=1}^{N} \Re\lambda_j |u_j|^2 \leq \| \sum_{j=1}^{N} u_j X_j \|^2_{H^1(I)} \leq c_2 \sum_{j=1}^{N} \Re\lambda_j |u_j|^2 \,, \\ c_1 \sum_{j=1}^{N} |u_j|^2 \leq \| \sum_{j=1}^{N} u_j X_j \|^2_{L^2(I)} \leq c_2 \sum_{j=1}^{N} |u_j|^2. \end{aligned} \tag{3.15}$$

∎

Proof: Let $(\lambda_j, X_j), j = 1, \ldots$, be the eigenpairs of system (3.2) with eigenfunctions normalized in $H^1(I)$, $\|X_j\|_{H^1(I)} = 1$. By Proposition 1, the (negative) operator is dissipative, its eigenvalues are simple, $|\lambda_j| \to \infty$, and satisfy the Glazman criterion (2.2). By the Glazman Theorem the eigenfunctions constitute a Riesz basis for the closure of their span, and by the Keldyš Theorem the closure equals the whole space $H^1(I)$. In particular, we have

$$c_1 \sum_{j=1}^{N} |u_j|^2 \leq \| \sum_{j=1}^{N} u_j X_j \|^2_{H^1(I)} \leq c_2 \sum_{j=1}^{N} |u_j|^2 \tag{3.16}$$

for some $c_1, c_2 > 0$, uniformly in N.

We need a corresponding estimate for the L^2-norm. Recall the eigenvalue problem reformulated in the closed operator setting (1.7). Applying A^{-1} gives

$$\begin{cases} u \in L^2(I) \setminus \{0\}, \ \lambda \in \mathbb{C} \\ (I + A^{-1})u = (\underbrace{\frac{1}{\lambda} + 1}_{=:\mu})u\,. \end{cases}$$

Let us check the Glazman criterion for the μ eigenvalues:

$$\begin{aligned} \Im\mu_j &= \Im(\tfrac{1}{\lambda_j} + 1) = \Im\tfrac{1}{\lambda_j} = -\tfrac{\Im\lambda_j}{|\lambda_j|^2}, \\ |\mu_j - \overline{\mu_k}| &= |\tfrac{1}{\lambda_j} - \tfrac{1}{\overline{\lambda_k}}| = \tfrac{|\lambda_j - \overline{\lambda_k}|}{|\lambda_j||\lambda_k|}, \\ \sum_{j\neq k} \frac{\Im\mu_j \Im\mu_k}{|\mu_j - \overline{\mu_k}|^2} &= \sum_{j\neq k} \frac{\Im\lambda_j \Im\lambda_k}{|\lambda_j - \overline{\lambda_k}|^2} < \infty\,. \end{aligned}$$

Tapping thus once again into the Glazman and Keldyš results, we find that the same eigenbasis, but now normalized in the L^2-norm, is also a Riesz basis in $L^2(I)$. This implies that

$$\begin{aligned} d_1 \sum_{j=1}^N |u_j|^2 \|X_j\|^2_{L^2(I)} \leq \|\sum_{j=1}^N u_j X_j\|^2_{L^2(I)} &= \Big\|\sum_{j=1}^N u_j \|X_j\|_{L^2(I)} \frac{X_j}{\|X_j\|_{L^2(I)}}\Big\|^2_{L^2(I)} \\ &\leq d_2 \sum_{j=1}^N |u_j|^2 \|X_j\|^2_{L^2(I)} \end{aligned} \tag{3.17}$$

with some other constants $d_1, d_2 > 0$, uniformly in N.

Switching to L^2-normalized eigenfunctions, estimates (3.16) and (3.17) read

$$\begin{aligned} c_1 \sum_{j=1}^N |u_j|^2 \|X_j\|^2_{H^1(I)} \leq \|\sum_{j=1}^N u_j X_j\|^2_{H^1(I)} &\leq c_2 \sum_{j=1}^N |u_j|^2 \|X_j\|^2_{H^1(I)}, \\ d_1 \sum_{j=1}^N |u_j|^2 \leq \|\sum_{j=1}^N u_j X_j\|^2_{L^2(I)} &\leq d_2 \sum_{j=1}^N |u_j|^2\,. \end{aligned}$$

We obtain the stated inequalities by noting that

$$\|X_j\|^2_{H^1(I)} = \Re\lambda_j \|X_j\|^2_{L^2(I)} = \Re\lambda_j. \tag{3.18}$$

∎

4 Stability of the Waveguide Problem

We are now in a position to prove the stability of the waveguide problem with impedance boundary condition on rectangular domains, as claimed in (1.2).

THEOREM 5
If p, u solve the waveguide problem (1.1) *on* $\Omega = (0,a) \times (0,l)$ *with fixed* $\omega, a > 0$ *and data* $f, g \in L^2(\Omega)$, *then*

$$\|p\|_{L^2(\Omega)} + \|u\|_{(L^2(\Omega))^2} \le C\, l\, (\|f\|_{L^2(\Omega)} + \|g\|_{(L^2(\Omega))^2})$$

holds with a constant $C > 0$ *that does not depend on* f, g, *and* l *(but possibly dependent on* ω*).* ∎

Proof: It is enough to bound $\|p\|_{H^1(\Omega)} \lesssim l\,\|r\|_{H^1(I)}$ for the Riesz representation r of the right-hand side functional, cf. (1.4). We denote the eigenpairs of (3.2) by (λ_j, X_j), $j = 1, \dots$ with $\|X_j\|_{L^2(I)} = 1$. Recall that $\lambda_j = \omega^2 - \beta_j^2 + 1$, cf. (1.6). By (3.6) of Lemma 2,

$$(\Re\lambda_j)^2 \le |\lambda_j|^2 \lesssim (\Re\lambda_j)^2. \tag{4.1}$$

By Proposition 2, we can expand $p(x,z) = \sum_j p_j(z) X_j(x)$ for each z. Then using the estimates of Proposition 2 and the stability inequality (1.8),

$$\begin{aligned}
\|p\|^2_{H^1(\Omega)} &= \|\sum_j p_j X_j\|^2_{H^1(\Omega)} \\
&= \int_0^l \left\{ \|\sum_j p_j X_j\|^2_{H^1(I)} + \|\sum_j p'_j X_j\|^2_{L^2(I)} \right\} dz \\
&\lesssim \sum_j \left\{ \int_0^l \Re\lambda_j |p_j(z)|^2\, dz + \int_0^l |p'_j(z)|^2\, dz \right\} && \text{(by (3.15))} \\
&\lesssim l^2 \sum_j \left\{ \int_0^l |(\frac{\partial r}{\partial z}, \bar{X}_j)_{L^2(I)}|^2\, dz + \frac{1}{\Re\lambda_j} \int_0^l |(r, \bar{X}_j)_{H^1(I)}|^2\, dz \right\} && \text{(by (1.8) and (4.1))} \\
&= l^2 \sum_j \left\{ \int_0^l |(\frac{\partial r}{\partial z}, \bar{X}_j)_{L^2(I)}|^2\, dz + \int_0^l |(r, \frac{\bar{X}_j}{\|\bar{X}_j\|_{H^1(I)}})_{H^1(I)}|^2\, dz \right\} && \text{(by (3.18))} \\
&= l^2 \int_0^l \left\{ \sum_j |(\frac{\partial r}{\partial z}, \bar{X}_j)_{L^2(I)}|^2 + \sum_j |(r, \frac{\bar{X}_j}{\|\bar{X}_j\|_{H^1(I)}})_{H^1(I)}|^2 \right\} dz \\
&\lesssim l^2 \int_0^l \left\{ \left\| \sum_j (\frac{\partial r}{\partial z}, \bar{X}_j)_{L^2(I)} X_j \right\|^2_{L^2(I)} \right. \\
&\qquad \left. + \left\| \sum_j (r, \frac{\bar{X}_j}{\|\bar{X}_j\|_{H^1(I)}})_{H^1(I)} \frac{X_j}{\|X_j\|} \right\|^2_{H^1(I)} \right\} && \text{(again by (3.15))} \\
&= l^2 \int_0^l \left\{ \|\frac{\partial r}{\partial z}\|^2_{L^2(I)} + \|r\|^2_{H^1(I)} \right\} dz && \text{(see (4.2) below)} \\
&= l^2 \|r\|^2_{H^1(\Omega)}\,.
\end{aligned}$$

Here we have used the biorthogonality of the eigenfunctions X_j and the corresponding eigenfunctions of the adjoint operator - $\bar{X}_j$, in both $H^1(I)$ and $L^2(I)$ inner products. Consequently,

$$\frac{\partial r}{\partial z} = \sum_j (\frac{\partial r}{\partial z}, \overline{X}_j)_{L^2(I)} X_j, \qquad r = \sum_j (r, \frac{\overline{X}_j}{\|X_j\|_{H^1(I)}})_{H^1(I)} \frac{X_j}{\|X_j\|_{H^1(I)}}. \tag{4.2}$$

We used these expansions in the penultimate step above. $\blacksquare$

5 Conclusions

In the paper, we analyzed the well-posedness of a 2D model acoustic waveguide with an impedance BC. We have extended the stability analysis from [9] proving that the operator is bounded below with a constant inversely proportional to the length l of the waveguide. The same techniques can be used to show that the adjoint operator is also bounded below. Consequently, by the Closed Range Theorem, both the discussed problem and its adjoint are well-posed. The work constitutes a first step towards analyzing bent optical fibers.

Acknowledgments

L. Demkowicz and J. Gopalakrishnan were supported by AFOSR grant FA9550-23-1-0103 and NSF grant DMS-2245077. N. Heuer was supported by a JTO fellowship and ANID Fondecyt project 1230013.

References

[1] N. K. Bari. “Biorthogonal systems and bases in Hilbert space”. In: *Moskov. Gos. Univ. Učenye Zapiski Matematika* 148/4 (1951), pp. 69–107.

[2] L. Demkowicz and J. Gopalakrishnan. “A Class of Discontinuous Petrov-Galerkin Methods. Part II: Optimal Test Functions”. In: *Numer. Methods Partial Differ. Equ.* 27 (2011). See also ICES Report 2009-16, pp. 70–105.

[3] L. Demkowicz, J. Gopalakrishnan, I. Muga, and J. Zitelli. “Wavenumber Explicit Analysis for a DPG Method for the Multidimensional Helmholtz Equation”. In: *Comput. Methods Appl. Mech. Engrg.* 213-216 (2012), pp. 126–138.

[4] L. Demkowicz, M. Melenk, J. Badger, and S. Henneking. “Stability Analysis for Acoustic and Electromagnetic Waveguides. Part 2: Non-homogeneous Waveguides.” In: *Advances in Computational Mathematics* (2024). accepted, see also Oden Institute Report 2023/3.

[5] I. M. Glazman. “On expansibility in a system of eigenelements of dissipative operators”. In: *Uspehi Mat. Nauk (N.S.)* 13.3(81) (1958), pp. 179–181.

[6] I. Gohberg and M. Krein. *Introduction to the Theory of Linear Nonselfadjoint Operators in Hilbert Space.* Providence: American Mathematical Society, 1965.

[7] M. Halla. “On the existence and stability of modified Maxwell Steklov eigenvalues”. In: *SIAM J. Math. Anal.* 55.5 (2023), pp. 5445–5463.

[8] M. V. Keldyš. “On the characteristic values and characteristic functions of certain classes of non-self-adjoint equations”. In: *Doklady Akad. Nauk SSSR (N.S.)* 77 (1951), pp. 11–14.

[9] M. Melenk, L. Demkowicz, and S. Henneking. *Stability Analysis for Electromagnetic Waveguides. Part 1: Acoustic and Homogeneous Electromagnetic Waveguides.* Tech. rep. 2. submitted to SIAM Journal on Mathematical Analysis. The University of Texas at Austin, Austin, TX 78712, 2023.

[10] J. T. Schwartz. "Perturbations of spectral operators, and applications. I. Bounded perturbations." In: *Pacific Journal of Mathematics* 4 (1954), pp. 415–458.