GENERALIZING RIEMANN CURVATURE TO REGGE METRICS

JAY GOPALAKRISHNAN, MICHAEL NEUNTEUFEL, JOACHIM SCHÖBERL, AND MAX WARDETZKY

Abstract. In this paper we propose a generalization of the Riemann curvature tensor on manifolds (of dimension two or higher) endowed with a Regge metric. Specifically, while all components of the metric tensor are assumed to be smooth within elements of a triangulation of the manifold, they need not be smooth across element interfaces, where only continuity of the tangential components are assumed. While linear derivatives of the metric can be generalized as Schwartz distributions, similarly generalizing the classical Riemann curvature tensor, a nonlinear second-order derivative of the metric, requires more care. We propose a generalization combining the classical angle defect and jumps of the second fundamental form across element interfaces, and rigorously prove correctness of this generalization. Specifically, if a piecewise smooth metric approximates a globally smooth metric, our generalized Riemann curvature tensor approximates the classical Riemann curvature tensor arising from a globally smooth metric. Moreover, we show that if the metric approximation converges at some rate in a piecewise norm that scales like the L^2-norm, then the curvature approximation converges in the H^{-2}-norm at the same rate, under additional assumptions. By appropriate contractions of the generalized Riemann curvature tensor, this work also provides generalizations of scalar curvature, the Ricci curvature tensor, and the Einstein tensor in any dimension.

Keywords: Riemann curvature, finite element method, Regge calculus, differential geometry, Ricci curvature.

MSC2020: 65N30, 53A70, 83C27.

1. Introduction

Regge calculus, introduced by Tullio Regge in the 1960s [34], is now a firmly established technique in numerical relativity. It employs piecewise flat simplicial complexes to discretize the metric tensor of an N-dimensional manifold ($N \geqslant 2$) via edge-length specifications and discretizes curvatures via angle defects (also called angle deficits). These angle defects, which are concentrated at $(N-2)$-subsimplices, were shown [10] to converge to the scalar curvature in the sense of measures as the triangulation becomes finer. However, high-order approximations of the full Riemann curvature tensor in N dimensions has remained elusive so far. This work provides such an extension, and a rigorous proof of convergence in a Sobolev norm (including rates of convergence), confirming its accuracy across all dimensions. The heart of the matter goes beyond utilitarian considerations in numerically approximating curvature. In fact, our central contribution is the identification and the analysis of a generalized notion of Riemannian curvature on triangulated manifolds with isometrically glued simplices, whose intrinsic metric is not necessarily flat.

Regge's edge-length prescription is equivalent to defining a constant (flat) metric tensor within each N-simplex ("element") of the simplicial complex such that their tangential components are continuous across element interfaces [38, § II.A]. By using higher-order polynomial approximations of the metric tensor, rather than just constants, within each N-simplex, while maintaining the same tangential inter-element continuity, one can generate better metric approximations on the simplicial complex. Such piecewise polynomial metrics, possessing inter-element tangential continuity, form our point of departure. We refer to such metrics as "Regge metrics". Since these metrics are smooth within each element, one can view each element, by itself, as a Riemannian manifold with boundary. Imposing tangential continuity of the metric tensor across elements, precisely defined in §2.1 and referred to as "*tt*-continuity" throughout this paper, is tantamount to isometrically gluing adjacent element manifolds.

Two important questions arise. How can one craft a notion of Riemannian curvature from such a (possibly discontinuous) Regge metric? Will such a generalized notion of Riemannian curvature produce good approximations to classical Riemannian curvature if the Regge metric approximates a globally smooth metric? Let us start by outlining our approach to answering the first question. Obviously, in the interior of each N-simplex, classical formulas for the Riemann curvature tensor apply. These curvature contributions from the interior must be added to the angle defects, which provide curvature contributions concentrated at $(N-2)$-subsimplices ("bones"). We show that an additional type of curvature contributions is required, namely curvature contributions concentrated at $(N-1)$-subsimplices ("facets"). These contributions correspond to the jump of the second fundamental form across facets. We prove that this additional contribution suffices in order to obtain a generalization of the Riemann curvature tensor, for which high-order convergence rates can be established, when the Regge metric approximates a given smooth metric.

In two dimensions, the full Riemann curvature tensor, which encodes all intrinsic curvature information, reduces to the classical Gauss curvature. Numerous prior works have proposed and analyzed a generalization of Gauss curvature. For example, in [7, 20, 23, 39], the authors considered three types of curvature contributions: An element contribution obtained by classical Gauss curvature formulas, a bone contribution at each vertex of the simplicial complex consisting of the angle deficit, and a facet contribution at each edge consisting of the jump of geodesic curvature. The latter can be transparently motivated by the Gauss–Bonnet theorem. Since the geodesic curvature (κ) of each element's edge is related to the second fundamental form ($\mathrm{I\!I}$) by $\kappa = \mathrm{I\!I}(X, X)$ for any unit tangent vector X along the edge, our generalized curvature in N dimensions reduces to the known two-dimensional construction (see §2.6). Nevertheless, since the Gauss–Bonnet theorem does not apply in higher dimensions, we now provide more motivation and geometric insight for our proposed facet contribution of curvatures.

Consider a facet F that is a subsimplex of two adjacent elements $T_\pm$ in the simplicial complex with respective inward unit normal vectors $\hat{\nu}_\pm$ at a point p in F. Also consider two 2-dimensional submanifolds $S_\pm$ of $T_\pm$ passing through p and transversely intersecting F along a smooth curve γ in F, such that one tangent direction at p is parallel to the respective normal $\hat{\nu}_\pm$. Denote the unit tangent vector along γ at p by X. Let $\mathcal{R}_\pm$ be the respective Riemann curvature tensors of $T_\pm$. The respective sectional curvatures at p generated by the orthonormal pair

$X, \hat{\nu}_\pm$ equal $\mathcal{R}_\pm(X, \hat{\nu}_\pm, \hat{\nu}_\pm, X)$. It is well-known that these sectional curvatures coincide with the Gauss curvatures of $S_\pm$. Since we know, from the aforementioned two-dimensional result, that the generalized Gauss curvature of the composite 2-manifold $S_+ \cup S_-$ must have an interface curvature contribution arising from the difference of the geodesic curvatures $\kappa_\pm \equiv \mathbb{I}_\pm(X, X)$ from adjacent elements, we conclude that the difference between $\mathcal{R}_\pm(X, \hat{\nu}_\pm, \hat{\nu}_\pm, X)$ at p must equal the difference between $\mathbb{I}_+(X, X)$ and $\mathbb{I}_-(X, X)$. This result can be extended to any X in the tangent space of F at p by varying the 2-manifold. Furthermore, by the symmetry of $\mathcal{R}_\pm(X, \hat{\nu}_\pm, \hat{\nu}_\pm, Y)$ in X and Y, the polarization identity implies that the jump of $\mathcal{R}_\pm(X, \hat{\nu}_\pm, \hat{\nu}_\pm, Y)$ must equal the jump of $\mathbb{I}_\pm(X, Y)$ for any X, Y in the tangent space of F. Thus, a generalization of the Riemann curvature tensor must include a facet contribution arising as a jump of the second fundamental form at facets. Further motivation will be provided in Remark 2.1. Of course, no amount of motivation will replace the rigorous proof that we provide in this article.

One central contribution of our paper is the convergence analysis of this generalized notion of Riemannian curvatures. Our analysis relies on the *linearization* of Riemannian curvature and of the other terms in the generalized Riemann curvature we propose. In two dimensions, as shown in [23], linearization of Gauss curvature leads to the so-called incompatibility operator, a second-order linear differential operator involving two curls, which also arises in elasticity when quantifying violation of St. Venant's compatibility conditions. We provide a corresponding linearization of the Riemann curvature tensor in higher dimensions. Similar linearizations have also been considered in the literature of Ricci flows [40]. Comparing the linearization to the known incompatibility operators in two and three dimensions, we are led to a generalization of the covariant incompatibility operator in arbitrary dimensions (named "Inc" and defined in (4.4)). Its adjoint, generalized to piecewise smooth metrics, turns out to be a crucial ingredient in our analysis. We often refer to our generalized Riemann curvature as a "distributional Riemann curvature" (although it is technically not a distribution) because its linearization around a flat metric gives terms that correspond exactly to those of a distributional version of the linear incompatibility operator. Another key idea in our analysis is a variant of the Uhlenbeck trick [24, 40]. Namely, when a Regge metric approximates a smooth metric, comparing geometric quantities that depend on the metric is made easier if their metric dependence can be transformed into a common metric-independent space. We identify such a space in §3.1.

Related prior results. Comprehensive overviews of the development and impact of Regge calculus over the past fifty years, with broad applications in relativity and quantum mechanics, can be found in the works of [42, 35, 6]. As previously mentioned, the first rigorous proof demonstrating the convergence of Regge's angle deficit to the scalar curvature, within a sequence of appropriate triangulations in the sense of measures, was achieved in [10]. Subsequently, Christiansen demonstrated in [14, 15] that, for a given metric in the lowest-order Regge finite element space, the curvature of a sequence of mollified metrics converges to the angle deficit in the sense of measures. Methods rooted in angle deficit for approximating Gauss curvature on triangulations composed of piecewise flat triangles are well-established in Discrete Differential Geometry (DDG) and computer graphics. While convergence in the L^∞-norm up to quadratic order has been proven [9, 43, 44] on specific triangulations satisfying certain conditions, convergence is not guaranteed on general irregular grids. In [30], Regge's concept of angle deficit was extended to quadrilateral meshes. Noteworthy among the results applicable to higher-dimensional manifolds is the proof of convergence for approximated Ricci curvatures of isometrically embedded hypersurfaces $\subset \mathbb{R}^{N+1}$, as presented in [18] and later utilized for Ricci flows in [19].

Another perspective to contextualize the contemporary advancements in Regge finite elements is within the framework of *finite element exterior calculus* (FEEC) [4, 3]. The recognition of the utility of discrete spaces featuring constant metric tensors with continuous tangential-tangential components dates back to Sorkin [38]. Christiansen introduced subsequent developments in finite element structures for Regge calculus in [12], and these elements gained popularity in FEEC under the designation "Regge finite elements" [29]. Regge elements approximating metric and strain tensors were further extended to arbitrary polynomial orders on triangles, tetrahedra, and higher-dimensional simplices in [29], as well as quadrilaterals, hexahedra, and prisms in [31]. The effectiveness of Regge elements in discretizing portions of the Kröner complex or the elasticity complex was explored in [5, 13, 25]. Notably, properties of Regge elements were leveraged to devise a method circumventing membrane locking for general triangular shell elements in [32].

In [20], the Regge finite elements on 2-manifolds were used to develop a high-order Gauss curvature approximation. The key ingredient was an integral representation of the angle deficit extended to high-order. This formulation, which can be seen as a covariant version of the Hellan–Herrmann–Johnson (HHJ) method [16], enabled rigorous proofs of convergence at specific rates. This approach was reformulated in [7] in terms of a nonlinear distributional Gauss curvature, consisting of elementwise Gauss curvature, jumps of geodesic curvature at edges, and angle deficit at vertices as sources of curvature—see also [39] for a derivation of a curvature notion on singular surfaces in the sense of measures. Under the assumption that the metric approximation is produced by the canonical Regge interpolation operator, an improved convergence for the distributional Gauss curvature was proven in [23]. The first extension of distributional curvatures in dimension $N \geqslant 3$ has been proposed in [21], where the distributional scalar curvature has been defined and analyzed in any dimension. Therein the jump of the mean curvature at codimension 1 facets and the angle deficit at codimension 2 boundaries has been used as additional sources of curvature. Terms only up to codimension 2 boundaries are considered, which reflects the fact that second order derivatives of the metric tensor are needed to compute curvatures. Further, in [21] the H^{-2}-norm has been taken to measure the approximation error. The analysis there showed that in $N = 2$ convergence rates of $\mathcal{O}(h^{k+1})$ for Regge metrics of polynomial order $k \geqslant 0$ are obtained, whereas for $N \geqslant 3$ approximation $k \geqslant 1$ is necessary. Here, h denotes the meshsize of the triangulation. A significant difference between dimensions $N = 2$ and $N \geqslant 3$ is the appearance of a second part of the integral representation besides the covariant HHJ bilinear form. This second part, being responsible for no convergence in the lowest-order case $k = 0$, has been identified as the distributional Einstein tensor [22], extending the classical one to non-smooth Regge metrics.

Outline of contributions. In Section 2, we motivate and define a generalized Riemann curvature for any given *tt*-continuous Regge metrics on an N-manifold. We show that our definition reduces to the distributional Gauss curvature of [7, 23] when $N = 2$ in §2.6. Prior notions of distributional scalar curvature [21] and Einstein tensor [22] are shown to be recoverable from our generalized Riemann curvature in §2.7 and §2.9, respectively. A generalized Ricci tensor for Regge metrics in arbitrary dimension is derived in §2.8, which to our knowledge has not appeared in prior literature.

The main convergence results for the generalized Riemann curvature are in Section 5, specifically, Theorems 5.2 and Corollary 5.3. We show that if a sequence g_h of Regge metrics interpolating a smooth metric g using piecewise polynomials of order $k \geqslant 0$ in $N = 2$ ($k \geqslant 1$ in $N \geqslant 3$) is given, then the generalized Riemann curvature produced by g_h converges to the exact smooth Riemann curvature of g at the rate $\mathcal{O}(h^{k+1})$ in the H^{-2}-norm. For proving this, we combine ideas from [23] and [21], using the new ingredients made in previous sections, described next.

A central new ingredient, found in Section 3, is the linearization of the distributional Riemann curvature tensor as the underlying metric changes. The generalized curvature is given in Section 2 using geometrically natural test functions having the same symmetries as the Riemann curvature tensor. However, these test functions are metric-dependent, creating difficulties in the analysis by generating opaque metric dependencies. To circumvent these difficulties, we develop a version of the Uhlenbeck trick [24, 40], mapping test functions to certain metric independent ones, in §3.1. This then allows us to compute the linearization of all curvature contributions, thus proving the main result of the section, Theorem 3.20.

The result of Theorem 3.20 sets the stage for Section 4 where the incompatibility operator is defined in N dimensions. Specifically, Theorem 3.20 shows that the linearization consists of two parts. In Section 4, we show that one part of the linearization can be interpreted as the distributional covariant incompatibility operator, as in [23], but now extended to any dimension, as shown in Theorem 4.1. The remaining part consist of terms without covariant derivatives of the metric perturbation. The latter is zero in dimension $N = 2$, so never appeared in [23]. The final important ingredient is the adjoint of the distributional covariant incompatibility operator, the subject of Theorem 4.7. Each of these ingredients, interesting by itself, are used in subsequent section (Section 5) on numerical analysis which proceeds by estimating each of the above-mentioned two parts of the linearization.

In Section 6, we simplify and specialize the distributional Riemann curvature tensor in $N = 2, 3$ dimensions. When $N = 2$, we obtain the distributional Gauss curvature as expected, and we highlight the peculiarity where one of the two parts in the linearization vanishes. When $N = 3$, we focus on a curvature operator $\mathcal{Q}$ that encapsulates the (skew-) symmetry properties of the Riemann curvature. All terms are then written out in a computable form.

The computable form is leveraged in Section 7 where a numerical example is displayed in the $N = 3$ case. The results provide a practical illustration of asymptotic theoretical convergence rates. They also indicate that the rates proved are likely not improvable in general. In the lowest-order case of a piecewise constant metric tensor, we observe a large pre-asymptotic regime of linear convergence, which then eventually appear to degenerate to no convergence, as expected from the theory.

2. Distributional densitized Riemann curvature tensor

Consider an open domain $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, on which a smooth metric $\bar{g}$ provides a Riemannian manifold structure. We are not given $\bar{g}$, only an approximation of it, denoted by g. This approximation g is a piecewise smooth metric with respect to a triangulation $\mathscr{T}$. We assume that Ω can be subdivided into finitely many bounded elements which are diffeomorphisms of N-simplices and these curved elements are collected into $\mathscr{T}$. The goal of this section is to generalize the notion of Riemann curvature for metrics g that are only piecewise smooth, specifically for g in a Regge space defined in (2.4) below.

A foreword on notation is in order. The symbols $\mathfrak{X}(T)$, $\wedge^k(T)$, and $\mathcal{T}^k_l(T)$ denote, respectively, the spaces of smooth vector fields, k-form fields, and (k,l)-tensor fields on a submanifold T of Ω. Here smoothness signifies not only infinite differentiability at interior points but also smoothness up to (including) the boundary of T. In such symbols, replacement of the manifold T by a collection of subdomains such as the triangulation $\mathscr{T}$, signifies the piecewise smooth analogue with respect to the collection. Namely, $\mathcal{T}^k_l(\mathscr{T})$ is the Cartesian product of $\mathcal{T}^k_l(T)$ over some enumeration of all $T \in \mathscr{T}$. Also, $\wedge^1(\mathscr{T}) = \mathcal{T}^1_0(\mathscr{T})$ and $\mathfrak{X}(\mathscr{T}) = \mathcal{T}^0_1(\mathscr{T})$. Our metric g is in $\mathcal{T}^2_0(\mathscr{T})$, but may not be in $\mathcal{T}^2_0(\Omega)$ generally. Let $\triangle_{-m}T$ denote the set of $(N-m)$-dimensional subsimplices of an N-simplex T. The collection of all subsimplices in $\triangle_{-1}T$ for all $T \in \mathscr{T}$ is the set $\mathscr{F}$ of facets of $\mathscr{T}$. For an element $T \in \mathscr{T}$ and any $F \in \triangle_{-1}T$, we shall refer to a vector field $\hat\nu$ satisfying

$$g|_T(\hat\nu, X) = 0 \text{ for all } X \in \mathfrak{X}(F) \quad \text{and} \quad g|_T(\hat\nu,\hat\nu) = 1, \tag{2.1}$$

as a *g-normal vector of F in T*. There are two possible orientations of $\hat\nu$ (and we will select one when needed). For vector fields X on T on F, let

$$QX = X - g|_T(X,\hat\nu)\hat\nu \tag{2.2}$$

denote the pointwise tangential projection onto the tangent bundle of F. When restricting k-linear forms $B(X_1,\ldots,X_k)$ to subsimplices, we often need to consider cases where some arguments are fixed to some specific vector, while other arguments are restricted to tangent vector fields on the subsimplex. For example, on an $F \in \triangle_{-1}T$, given smooth vector fields X_i on T or F, the forms $B_{\hat\nu F\cdots F}$ and $B_{F\hat\nu F\cdots F}$ are defined by

$$\begin{aligned} B_{\hat\nu F\cdots F}(X_1,X_2,\ldots,X_{k-1}) &= B(\hat\nu, QX_1,\ldots,QX_{k-1}),\\ B_{F\hat\nu F\cdots F}(X_1,X_2,\ldots,X_{k-1}) &= B(QX_1,\hat\nu, QX_2,\ldots,QX_{k-1}), \end{aligned} \tag{2.3a}$$

and this notation is extended to cases when more than one argument is fixed with $\hat\nu$ in the obvious fashion. When all arguments of B are projected by Q, we abbreviate the resulting form $B_{F\ldots F}$ to simply B_F. When an argument is left open, and not projected to the $\mathfrak{X}(F)$, then we indicate so by a subscript "$\cdot$" in place of F, e.g., for a form $B(X,Y,Z,W)$ taking four arguments, the form $B_{\hat\nu FF\cdot}$ takes three arguments and is defined by

$$B_{\hat\nu FF\cdot}(X_1,X_2,X_3) = B(\hat\nu, QX_1, QX_2, X_3).$$

Obvious extensions of this notation to vectors other than $\hat\nu$ and to lower dimensional subsimplices are used without much ado.

We also use standard notions from smooth Riemannian geometry throughout, some of which are collected in Appendix A for ready reference.

2.1. **The *tt*-continuity.** Let $\mathcal{S}(\mathscr{T}) = \{\sigma \in \mathcal{T}^2_0(\mathscr{T}) : \sigma(X,Y) = \sigma(Y,X) \text{ for } X,Y \in \mathfrak{X}(\mathscr{T})\}$ be the space of symmetric covariant 2-tensors on Ω with no interelement continuity in general. With $\mathcal{S}^+(\mathscr{T}) = \{\sigma \in \mathcal{S}(\mathscr{T}) : \sigma(X,X) > 0 \text{ for all } 0 \neq X \in \mathfrak{X}(\mathscr{T})\}$ we denote the subspace of positive definite 2-tensors. Recalling that $\mathscr{F}$ denotes the set of all mesh facets (of codimension 1), we divide $\mathscr{F}$ into $\mathscr{F}_\partial$ consisting of facets contained in $\partial\Omega$ and the remainder, the set of interior facets, $\mathring{\mathscr{F}}$. Every

$F \in \mathring{\mathscr{F}}$ is of the form $F = \triangle_{-1}T_+ \cap \triangle_{-1}T_-$ for two elements $T_\pm \in \mathscr{T}$. We say that a $\sigma \in \mathcal{S}(\mathscr{T})$ has "tangential-tangential continuity" or *"tt-continuity"* if $\sigma|_{T_+}(X,Y) = \sigma|_{T_-}(X,Y)$ for all tangential vector fields $X, Y \in \mathfrak{X}(F)$ for every F in $\mathring{\mathscr{F}}$ (i.e., $\sigma(X,Y)$ is single-valued on all $F \in \mathring{\mathscr{F}}$). This brings us to the *Regge space*

$$\operatorname{Reg}(\mathscr{T}) = \{\sigma \in \mathcal{S}(\mathscr{T}) : \sigma \text{ is } tt\text{-continuous}\}, \tag{2.4}$$

and its subset $\operatorname{Reg}^+(\mathscr{T}) = \operatorname{Reg}(\mathscr{T}) \cap \mathcal{S}^+(\mathscr{T})$. The approximate metric g is assumed to be in $\operatorname{Reg}^+(\mathscr{T})$. Also define

$$\mathring{\operatorname{Reg}}(\mathscr{T}) = \{\sigma \in \operatorname{Reg}(\mathscr{T}) : \ \sigma(X,Y) = 0 \text{ for all } X, Y \in \mathfrak{X}(F), F \in \mathscr{F}_\partial\}. \tag{2.5}$$

2.2. **Curvature within elements.** Since a g in $\operatorname{Reg}^+(\mathscr{T})$ is smooth within each element $T \in \mathscr{T}$, it generates a unique Levi–Civita connection in each T, denoted by ∇. We refer to Appendix A for a summary of used geometric notation. Using ∇, the Riemann curvature tensor within T can be computed using standard formulas: following the sign convention of [28], define

$$\mathcal{R}(X,Y,Z,W) = g(R_{X,Y}Z, W), \qquad X, Y, Z, W \in \mathfrak{X}(\mathscr{T}), \tag{2.6}$$

where $R_{X,Y}Z = \nabla_X\nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z$ is the Riemann curvature endomorphism. This element-by-element curvature tensor $\mathcal{R} \in \mathcal{T}^4_0(\mathscr{T})$ is only one of the contributors to the total distributional curvature defined below.

2.3. **Jump of the second fundamental form.** The jumps of g create further sources of curvatures at lower dimensional facets, which must be added to the curvature within elements to get good curvature approximations. Recall the g-normal vector $\hat\nu$ of (2.1). The second fundamental form [28] of F considered as an embedded submanifold depends on the orientation of $\hat\nu$ and is defined by

$$\mathrm{I\!I}^{\hat\nu}(X,Y) = -g(\nabla_X \hat\nu, Y) = g(\hat\nu, \nabla_X Y), \qquad X, Y \in \mathfrak{X}(F). \tag{2.7}$$

The second equality follows from differentiating the identity $g(\hat\nu, Z) = 0$ for $Z \in \mathfrak{X}(F)$. Let the unique g-normal vector on F that points *inward* into an element $T \in \mathscr{T}$ be denoted by $\hat\nu_F^T$. Now consider an $F = \triangle_{-1}T_+ \cap \triangle_{-1}T_-$ for some $T_\pm \in \mathscr{T}$. As $g \in \operatorname{Reg}^+(\mathscr{T})$ is solely tt-continuous, $\hat\nu_F^{T_+} \neq -\hat\nu_F^{T_-}$ in general. The *jump of the second fundamental form* across F is defined by

$$[\![\mathrm{I\!I}]\!](X,Y) = \mathrm{I\!I}^{\hat\nu_F^{T_+}}(X,Y) + \mathrm{I\!I}^{\hat\nu_F^{T_-}}(X,Y), \tag{2.8}$$

for all $X, Y \in \mathfrak{X}(F)$. This jump function $[\![\mathrm{I\!I}]\!]$ is in $\mathcal{T}^2_0(\mathring{\mathscr{F}})$ (which per previous notation, is the Cartesian product of $\mathcal{T}^2_0(F)$ over an enumeration of all F in $\mathring{\mathscr{F}}$). It will act as a source of curvature on facets.

The jump of a general $\hat\nu$-dependent tensor $B^{\hat\nu} \in \mathcal{T}^k_0(\mathscr{T})$ across F is defined by

$$[\![B]\!](X_1, X_2, \ldots) = B^{\hat\nu_F^{T_+}}(X_1, X_2, \ldots) + B^{\hat\nu_F^{T_-}}(X_1, X_2, \ldots), \qquad X_i \in \mathfrak{X}(F), \tag{2.9}$$

in analogy with (2.8).

Remark 2.1. The jump of the second fundamental form as a source of curvature along a hypersurface F can be motivated by the *Radial Curvature Equation*, *Tangential Curvature Equation*, and *Normal Curvature Equation* [33, Theorem 3.2.2, Theorem 3.2.4, and Theorem 3.2.5]. (These equations are also called *Ricci*, *Gauss*, and *Codazzi* equations [27].) They imply (after setting f in [33, Theorem 3.2.2] as the signed distance function of the hypersurface F) that

$$\begin{aligned}
\mathcal{R}(X,\hat\nu,\hat\nu,Y) &= (\nabla_{\hat\nu}\mathrm{II})(X,Y) - \mathrm{III}(X,Y),\\
\mathcal{R}(X,Y,Z,W) &= \mathcal{R}_F(X,Y,Z,W) - \mathrm{II}(X,W)\mathrm{II}(Y,Z) + \mathrm{II}(X,Z)\mathrm{II}(Y,W),\\
\mathcal{R}(X,Y,Z,\hat\nu) &= (\nabla_X\mathrm{II})(Y,Z) - (\nabla_Y\mathrm{II})(X,Z),
\end{aligned}$$

where $X,Y,Z,W \in \mathfrak{X}(F)$ are tangent vector fields, $\mathrm{III}(X,Y) = \langle \nabla_X\hat\nu, \nabla_Y\hat\nu\rangle$ denotes the third fundamental form, and $\mathcal{R}_F$ the Riemann curvature tensor on F. All other components of $\mathcal{R}$ can be traced back to the three above. Using a mollifier argument the normal derivative term $\nabla_{\hat\nu}\mathrm{II}$ produces the jump of the second fundamental form in the limit, whereas the jumps of the third fundamental form, $\mathcal{R}(X,Y,Z,W)$, and $\mathcal{R}(X,Y,Z,\hat\nu)$ vanish in the limit. Thus the jump in the Riemann curvature is completely characterized by the jump of the second fundamental form and

$$[\![\mathcal{R}]\!](X,\hat\nu,\hat\nu,Y) = [\![\mathrm{II}]\!](X,Y).$$

2.4. **Angle deficit.** There are also sources of curvature along subsimplices of codimension 2. Let $\mathscr{E}$ denote the collection of all simplices in $\triangle_{-2}T$ for all $T\in\mathscr{T}$. Divide it into $\mathscr{E}_\partial$ consisting of simplices in $\mathscr{E}$ lying on the boundary $\partial\Omega$ and the remainder $\mathring{\mathscr{E}} = \mathscr{E}\backslash\mathscr{E}_\partial$. Given any $E\in\mathring{\mathscr{E}}$ there is an $F\in\mathring{\mathscr{F}}$ such that $E\in\triangle_{-1}F$. We refer to a vector field $\hat\mu\in\mathfrak{X}(F)$ satisfying

(2.10)
$$g|_T(\hat\mu,\hat\nu_F^T) = 0,\quad F\in\triangle_{-1}T,\qquad g(\hat\mu,X) = 0 \text{ for all } X\in\mathfrak{X}(E),\qquad g(\hat\mu,\hat\mu) = 1,$$

as a *g-conormal vector of E in F*. There are two possible orientations for such a $\hat\mu$. When both E and F lie on the boundary of an element $T\in\mathscr{T}$, we select the unique g-conormal vector of E that points into F from E and denote it by $\hat\mu_E^F$ (see Figure 1). Using the two facets $F_\pm$ in $\triangle_{-1}T$ such that $E = \triangle_{-1}F_+\cap\triangle_{-1}F_-$, we define the following angle function on E

$$\sphericalangle_E^T = \arccos\left(g\big|_T(\hat\mu_E^{F_+},\hat\mu_E^{F_-})\right). \tag{2.11}$$

Let $\mathscr{T}_E = \{T\in\mathscr{T} : E\in\triangle_{-2}T\}$. The *angle deficit* at $E\in\mathring{\mathscr{E}}$ is defined by

$$\Theta_E = 2\pi - \sum_{T\in\mathscr{T}_E}\sphericalangle_E^T.$$

This function, $\Theta = \Pi_{E\in\mathring{\mathscr{E}}}\Theta_E \in \mathcal{T}_0^0(\mathring{\mathscr{E}})$, will act as a source of curvature on $\mathring{\mathscr{E}}$.

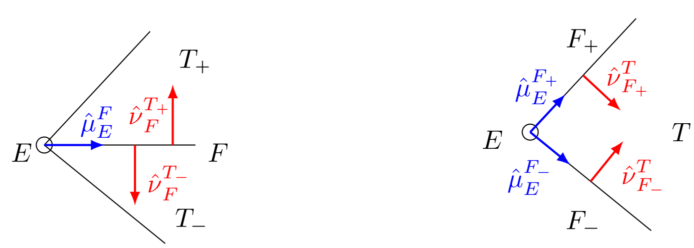

FIGURE 1. Visualization of g-normal and g-conormal vectors on facet between two elements (left) and a single element (right).

2.5. **Riemann curvature for Regge metrics.** Let $A \in \mathcal{T}^4_0(\mathscr{T})$ have the (skew) symmetries of the Riemann curvature tensor, i.e.,

$$A(X,Y,Z,W) = -A(Y,X,Z,W) = -A(X,Y,W,Z) = A(Z,W,X,Y) \tag{2.12}$$

for all $X,Y,Z,W \in \mathfrak{X}(\mathscr{T})$. We will generalize the Riemann curvature tensor as a linear functional acting on such A. We shall also require A to have certain interelement continuity constraints described next. Recall that per (2.3), $A_{F\hat{\nu}\hat{\nu}F} \in \mathcal{T}^2_0(\mathscr{F})$ is defined by

$$A_{F\hat{\nu}\hat{\nu}F}(X,Y) = A(X,\hat{\nu},\hat{\nu},Y), \qquad X,Y \in \mathfrak{X}(F),$$

for any F in $\mathscr{F}$ and a g-normal vector $\hat{\nu}$ as in (2.1). Note that $A_{F\hat{\nu}\hat{\nu}F}$ is independent of the orientation of $\hat{\nu}$. Further, due to (2.12), $A_{F\hat{\nu}\hat{\nu}F} = A_{\cdot\hat{\nu}\hat{\nu}\cdot}$.

In general, since limiting values of A from adjacent elements are different, $A_{F\hat{\nu}\hat{\nu}F}$ is discontinuous with multi-valued limits on element interfaces. We consider the following continuity requirement:

$$A_{F\hat{\nu}\hat{\nu}F} \text{ is single-valued for all } F \in \mathring{\mathscr{F}}. \tag{2.13}$$

Define the *test space* $\mathring{\mathcal{A}}$ by

$$\begin{aligned} \mathcal{A} :=& \{A \in \mathcal{T}^4_0(\mathscr{T}) \,:\, A \text{ satisfies (2.12) and (2.13)}\}, \\ \mathring{\mathcal{A}} :=& \{A \in \mathcal{A} \,:\, A_{F\hat{\nu}\hat{\nu}F} \text{ vanishes on all } F \in \mathscr{F}_\partial\}. \end{aligned} \tag{2.14}$$

The continuity condition (2.13) implies continuity at mesh interfaces of codimension two, as we now show.

Lemma 2.2. *Let $A \in \mathcal{A}$ and $E \in \mathring{\mathscr{E}}$. Let $F \in \mathring{\mathscr{F}}$ and $T \in \mathscr{T}$ be such that $E \in \triangle_{-1}F$ and $F \in \triangle_{-1}T$. If $\hat{\nu}$ is a g-normal vector of F in T (see* (2.1)*) and $\hat{\mu}$ is a g-conormal vector of E in F (see* (2.10)*), then $A_{\hat{\mu}\hat{\nu}\hat{\nu}\hat{\mu}} = A(\hat{\mu},\hat{\nu},\hat{\nu},\hat{\mu})$ is single-valued for all $E \in \mathring{\mathscr{E}}$.*

Proof. Let $T_\pm \in \mathscr{T}$ share a facet $F = \triangle_{-1}T_+ \cap \triangle_{-1}T_-$ with $E \in \triangle_{-1}F$. Let $\hat{\tau} \in \mathfrak{X}(F)$ denote some extension of $\hat{\mu}$ from E to F, i.e., $\hat{\tau}|_E = \hat{\mu}$. Because of (2.13), the values of $A(\hat{\tau},\hat{\nu},\hat{\nu},\hat{\tau})$ from T_+ and T_- are equal at any point on F. By the assumption that elements in $\mathcal{T}^4_0(\mathscr{T})$ are continuous up to the element boundaries, the same holds for points on E. Thus the values of $A_{\hat{\mu}\hat{\nu}\hat{\nu}\hat{\mu}}$ from T_+ and T_- coincide on E.

Next, let $F_\pm \in \triangle_{-1}T$ be such that $\triangle_{-1}F_+ \cap \triangle_{-1}F_- = E$. The proof will be finished if we show that at any point in E, the values of $A_{\hat{\mu}\hat{\nu}\hat{\nu}\hat{\mu}}$ from F_+ and F_- coincide, i.e., if $A(\hat{\mu}_E^{F_+},\hat{\nu}_{F_+}^T,\hat{\nu}_{F_+}^T,\hat{\mu}_E^{F_+})$ and $A(\hat{\mu}_E^{F_-},\hat{\nu}_{F_-}^T,\hat{\nu}_{F_-}^T,\hat{\mu}_E^{F_-})$ coincide. (Figure 1 illustrates the vectors involved.) Since $\{\hat{\nu}_{F_+}^T,\hat{\mu}_E^{F_+}\}$ and $\{\hat{\nu}_{F_-}^T,\hat{\mu}_E^{F_-}\}$ span the same 2-dimensional plane, there is an angle ϕ (measured in $g|_T$) by which $\hat{\mu}_E^{F_+}$ can be rotated in plane to $\hat{\mu}_E^{F_-}$. Then, noting that $\hat{\nu}_{F_+}^T$ and $\hat{\nu}_{F_-}^T$ point into T,

$$\hat{\mu}_E^{F_-} = \cos(\phi)\hat{\mu}_E^{F_+} + \sin(\phi)\hat{\nu}_{F_+}^T, \qquad \hat{\nu}_{F_-}^T = \sin(\phi)\hat{\mu}_E^{F_+} - \cos(\phi)\hat{\nu}_{F_+}^T.$$

Using the skew symmetries (2.12) of A,

$$\begin{aligned} A(\hat{\mu}_E^{F_-},\hat{\nu}_{F_-}^T,\hat{\nu}_{F_-}^T,\hat{\mu}_E^{F_-}) &= (\cos^2(\phi)+\sin^2(\phi))A(\hat{\mu}_E^{F_+},\hat{\nu}_{F_+}^T,\hat{\nu}_{F_-}^T,\hat{\mu}_E^{F_-}) \\ &= A(\hat{\mu}_E^{F_+},\hat{\nu}_{F_+}^T,\hat{\nu}_{F_+}^T,\hat{\mu}_E^{F_+}), \end{aligned}$$

i.e., at E, the value of $A_{\hat{\mu}\hat{\nu}\hat{\nu}\hat{\mu}}$ from F_+ and F_- coincide. ∎

Definition 2.3. We define the *generalized densitized Riemann curvature* for the non-smooth metric $g \in \mathrm{Reg}^+(\mathscr{T})$ to be the linear functional $\widetilde{\mathcal{R}\omega} : \mathring{\mathcal{A}} \to \mathbb{R}$ given by

$$\begin{aligned}\widetilde{\mathcal{R}\omega}(A) = &\sum_{T \in \mathscr{T}} \int_T \langle \mathcal{R}, A \rangle \omega_T + 4 \sum_{F \in \mathring{\mathscr{F}}} \int_F \langle [\![\mathrm{I\!I}]\!], A_{F\hat{\nu}\hat{\nu}F} \rangle \omega_F \\ &+ 4 \sum_{E \in \mathring{\mathscr{E}}} \int_E \Theta_E \, A_{\hat{\mu}\hat{\nu}\hat{\nu}\hat{\mu}} \, \omega_E \end{aligned} \tag{2.15}$$

for all $A \in \mathring{\mathcal{A}}$, where $\omega_T = \omega|_T, \omega_F = \omega|_F$, and $\omega_E = \omega|_E$ are the volume forms on T, F, and E, respectively, given by the Riemannian volume form $\omega_g \equiv \omega \in \mathcal{T}^N_0(\mathscr{T})$ generated by g. (We drop the subscript in ω_g when there can be no confusion on what metric is being used.) Here and throughout, $\langle \cdot, \cdot \rangle$ denotes the standard extension of the g-inner product to tensors (see Appendix A, (A.9)), $\hat{\nu}$ is a g-normal vector of F, and $\hat{\mu}$ is a g-conormal vector of E. By Lemma 2.2, the last term in (2.15) makes sense.

The test space $\mathring{\mathcal{A}}$ contains infinitely smooth compactly supported tensor functions on Ω. Hence, $\widetilde{\mathcal{R}\omega}(A)$ can be regarded as an extension of a Schwartz distribution or a measure on Ω. Due to the presence of the volume form, it is a *distribution density* [14] as in the title of this section. Equipping $\mathring{\mathcal{A}}$ with a topology in which the right-hand side of (2.15) is continuous is an interesting issue (see e.g., a similar issue in [23, Appendix A]). But this is not discussed further in this paper because it is not central to our main effort of proving the correctness of (2.15) through numerical analysis, where we will only examine convergence of $\widetilde{\mathcal{R}\omega}$ in the $H^{-2}(\Omega)$-norm.

Note that there are many g-dependent quantities in (2.15). When we need to explicitly show that dependence, we will write g as an argument, i.e., instead of $\widetilde{\mathcal{R}\omega}(A), \omega_T$, ω_F, ω_E, $\hat{\nu}$, and $\hat{\mu}$, we write $\widetilde{\mathcal{R}\omega}(g)(A)$, $\omega_T(g)$, $\omega_F(g)$, $\omega_E(g)$, $\hat{\nu}(g)$, and $\hat{\mu}(g)$, respectively. Even the test space $\mathring{\mathcal{A}}$ is g-dependent, an issue we discuss in more detail in §3.

2.6. **Specialization to two-dimensional Gauss curvature case.** In the case of 2D manifolds ($N = 2$), elements of the test space can be generated by scalar fields in

$$\mathcal{V}(\mathscr{T}) = \{u \in \textstyle\bigwedge^0(\mathscr{T}) : \; u \text{ is continuous on } \Omega\},$$
$$\mathring{\mathcal{V}}(\mathscr{T}) = \{u \in \mathcal{V}(\mathscr{T}) : u|_{\partial\Omega} = 0\},$$

by combining them with the Riemannian volume form ω. Let $v \in \mathring{\mathcal{V}}(\mathscr{T})$. The tensor field $A = -v \, \omega \otimes \omega$,

$$A(X, Y, Z, W) = -v \, \omega(X, Y) \, \omega(Z, W), \tag{2.16}$$

obviously satisfies the symmetries in (2.12). Moreover, since ω applied to any g-orthonormal frame yields ± 1, we have $\omega(\hat{\mu}, \hat{\nu})\omega(\hat{\nu}, \hat{\mu}) = -1$, so the continuity of v implies (2.13). Hence, $A \in \mathring{\mathcal{A}}$. We use this choice of A in (2.15). Then, a computation using (A.9) and $\omega = \sqrt{\det g} \; dx^1 \wedge dx^2$ shows that the first term on the right-hand side of (2.15) takes the form

$$\langle \mathcal{R}, A \rangle = \frac{4 R_{1221} \, v}{\det g} = 4K \, v, \tag{2.17}$$

where K is the Gauss curvature.

We proceed to the next term on the right-hand side of (2.15). It is easy to see that for the A in (2.16),

$$A_{F\hat{\nu}\hat{\nu}F} = v\,\hat{\tau}^\flat \otimes \hat{\tau}^\flat,$$

for any g-normalized tangent $\hat{\tau}$ to the one-dimensional element boundary ∂T. Note that $\hat{\tau}$ and $\hat{\mu}$ are collinear at each point. Therefore, by the second equality of (2.7), the geodesic curvature $\kappa^{\hat{\nu}} = g(\nabla_{\hat{\tau}}\hat{\tau}, \hat{\nu})$ of ∂T equals $\mathbb{I}^{\hat{\nu}}(\hat{\tau}, \hat{\tau})$. Thus, the jump of the geodesic curvature across $F = \triangle_{-1}T_+ \cap \triangle_{-1}T_-$, namely $[\![\kappa]\!] = \kappa^{\hat{\nu}_F^{T_+}} + \kappa^{\hat{\nu}_F^{T_-}}$, satisfies

$$\langle [\![\mathbb{I}]\!], A_{F\hat{\nu}\hat{\nu}F}\rangle = [\![\kappa]\!] v.$$

Finally, to relate to the last term on the right-hand side of (2.15) on the angle deficit, observe that

$$\Theta_E A_{\hat{\mu}\hat{\nu}\hat{\nu}\hat{\mu}} = \Theta_E\, v \tag{2.18}$$

since $A_{\hat{\mu}\hat{\nu}\hat{\nu}\hat{\mu}} = -\omega(\hat{\mu}, \hat{\nu})\omega(\hat{\nu}, \hat{\mu})v = v$. Combining (2.17)–(2.18) we find that

$$\frac{1}{4}\widetilde{\mathcal{R}\omega}(-v\omega \otimes \omega) = \sum_{T\in\mathscr{T}} \int_T Kv\,\omega_T + \sum_{F\in\mathring{\mathscr{F}}} \int_F [\![\kappa]\!] v\,\omega_F + \sum_{E\in\mathring{\mathscr{E}}} \Theta_E\, v(E). \tag{2.19}$$

The right-hand side above is exactly the two-dimensional *distributional densitized Gauss curvature* $\widetilde{K\omega}(v)$ previously analyzed in [7, 23].

2.7. **Specialization to scalar curvature in any dimension.** The scalar curvature in the smooth case is given by $S = \mathcal{R}_{ijkl}g^{il}g^{jk}$. Its generalized densitized version $\widetilde{S\omega}$ can be obtained from our generalized Riemann curvature, as shown next. The Kulkarni–Nomizu product $\owedge : \mathcal{T}^2_0(\Omega) \times \mathcal{T}^2_0(\Omega) \to \mathcal{T}^4_0(\Omega)$ is defined as

$$\begin{aligned}(h \owedge k)(X, Y, Z, W) := {} & h(X, W)k(Y, Z) + h(Y, Z)k(X, W)\\ & - h(X, Z)k(Y, W) - h(Y, W)k(X, Z)\end{aligned}$$

taking two $(2, 0)$-tensors resulting into a $(4, 0)$-tensor with the algebraic properties of the Riemann curvature tensor, i.e., $(h\owedge k)(X, Y, Z, W) = -(h\owedge k)(Y, X, Z, W) = (h \owedge k)(Y, X, W, Z)$ and $(h \owedge k)(X, Y, Z, W) = (h \owedge k)(Z, W, X, Y)$.

We can choose $\mathring{\mathcal{A}}$ to be of the form

$$A = g \owedge g\, v, \qquad v \in \mathring{\mathcal{V}}, \tag{2.20}$$

fulfilling the continuity conditions (2.13), as on all $F \in \mathring{\mathscr{F}}$ and $E \in \mathring{\mathscr{E}}$

$(g \owedge g)(X, \hat{\nu}, \hat{\nu}, Y) = 2(g(X, Y) - g(\hat{\nu}, X)g(\hat{\nu}, Y)) = 2\, g(X, Y)$ for all $X, Y \in \mathfrak{X}(F)$,

$(g \owedge g)(\hat{\mu}, \hat{\nu}, \hat{\nu}, \hat{\mu}) = 2.$

Inserting (2.20) into (2.15) yields due to the (skew-)symmetry properties of $\mathcal{R}$

$$\langle \mathcal{R}, A\rangle = 4\mathcal{R}_{ijkl}g^{il}g^{jk}v = 4Sv, \quad \langle [\![\mathbb{I}]\!], A_{F\hat{\nu}\hat{\nu}F}\rangle = 2[\![H]\!]v, \quad \Theta_E\, A_{\hat{\mu}\hat{\nu}\hat{\nu}\hat{\mu}} = 2\Theta_E v,$$

where S and $H^{\hat{\nu}} = \operatorname{tr}(\mathbb{I}^{\hat{\nu}})$ denote the scalar and mean curvature, respectively. Thus, we obtain, $v \in \mathring{\mathcal{V}}$,

$$\frac{1}{4}\widetilde{\mathcal{R}\omega}(g \owedge gv) = \sum_{T\in\mathscr{T}} \int_T Sv\,\omega_T + 2\sum_{F\in\mathring{\mathscr{F}}} \int_F [\![H]\!]\, v\,\omega_F + 2\sum_{E\in\mathring{\mathscr{E}}} \int_E \Theta_E v\,\omega_E,$$

which coincides with the definition of the *distributional densitized scalar curvature*, $\widetilde{S\omega}(v)$, in any dimension proposed in [21]. The factor 2 in the codimension 1 and 2 boundary terms is consistent with (2.19) as in two dimensions the scalar curvature is twice the Gauss curvature, $S = 2K$.

2.8. **Specialization to Ricci curvature in any dimension.** For smooth metrics, the Ricci curvature is a contraction of the Riemann curvature tensor given by $\mathrm{Ric}_{ij} = \mathcal{R}_{kijl} g^{kl}$. To develop its generalization to Regge metrics, consider test functions of the form

$$A = g \owedge \rho, \tag{2.21}$$

where ρ is any tensor in the subspace $\mathcal{W}(\mathscr{T})$ of $\mathrm{Reg}(\mathscr{T})$ with the additional "*nn*-continuity" (normal-normal continuity) property that $\rho_{\hat\nu\hat\nu} = \rho(\hat\nu, \hat\nu)$ is single-valued at all interior facets in $F \in \mathring{\mathscr{F}}$ and $\rho_{\hat\nu\hat\nu}$ vanishes on the boundary $\partial\Omega$. The *nn*- and *tt*-continuity together imply that the trace of ρ is single-valued at facets and (due to the smoothness assumption up to element boundaries) also single-valued at codimension 2 boundaries. Hence the A in (2.21) satisfies, for all $X, Y \in \mathfrak{X}(F)$ and $F \in \mathring{\mathscr{F}}$,

$$\begin{aligned} A(X, \hat\nu, \hat\nu, Y) &= \rho(X, Y) + g(Y, X)\rho(\hat\nu, \hat\nu) - g(\hat\nu, X)\rho(\hat\nu, Y) - g(\hat\nu, Y)\rho(\hat\nu, X) \\ &= \rho(X, Y) + g(X, Y)\rho_{\hat\nu\hat\nu} \\ A(\hat\mu, \hat\nu, \hat\nu, \hat\mu) &= \rho(\hat\mu, \hat\mu) + \rho(\hat\nu, \hat\nu) = \mathrm{tr}(\rho) - \mathrm{tr}(\rho_E), \end{aligned}$$

where ρ_E denotes the restriction of ρ to an $E \in \mathring{\mathscr{E}}$. We conclude that the A in (2.21) is in $\mathring{\mathcal{A}}$. Inserting such A into (2.15) leads to a novel definition of the *generalized densitized Ricci curvature tensor*

$$\begin{aligned} \widetilde{\mathrm{Ric}\,\omega}(\rho) = \frac{1}{4}\widetilde{\mathcal{R}\omega}(g \owedge \rho) = \sum_{T\in\mathscr{T}} \int_T \langle \mathrm{Ric}, \rho\rangle \omega_T + \sum_{F\in\mathring{\mathscr{F}}} \int_F \langle [\![\mathrm{I\!I}]\!], \rho + \rho_{\hat\nu\hat\nu} g\rangle \omega_F \\ + \sum_{E\in\mathring{\mathscr{E}}} \int_E \Theta_E \,(\rho_{\hat\nu\hat\nu} + \rho_{\hat\mu\hat\mu})\, \omega_E, \end{aligned}$$

a linear functional acting on any ρ in $\mathcal{W}(\mathscr{T})$.

2.9. **Specialization to the Einstein tensor.** Let $N \geqslant 3$. The Einstein tensor for smooth metrics is given using the above-mentioned Ricci curvature and scalar curvature by $G = \mathrm{Ric} - \frac{1}{2} S\, g$. Its generalization to Regge metrics can be seen from our Riemann curvature generalization as follows. Define the bijective algebraic operator $J : \mathcal{S}(\mathscr{T}) \to \mathcal{S}(\mathscr{T})$ by $J\rho = \rho - \frac{1}{2}\mathrm{tr}(\rho) g$. Then, using the vanishing trace Regge subspace defined in (2.5), test functions of the form

$$A = g \owedge J\rho, \qquad \rho \in \mathring{\mathrm{Reg}}(\mathscr{T}).$$

are in $\mathring{\mathcal{A}}$, because on all $F \in \mathring{\mathscr{F}}$, $X, Y \in \mathfrak{X}(F)$, and $E \in \mathring{\mathscr{E}}$,

$$\begin{aligned} A_{\cdot\hat\nu\hat\nu\cdot}(X, Y) &= (J\rho)(X, Y) + g(Y, X)(J\rho)_{\hat\nu\hat\nu} - g(\hat\nu, X)(J\rho)(\hat\nu, Y) - g(\hat\nu, Y)(J\rho)(\hat\nu, X) \\ &= \rho(X, Y) + g(X, Y)\rho_{\hat\nu\hat\nu} - g(X, Y)\,\mathrm{tr}(\rho) = \mathbb{S}_F(\rho)(X, Y), \\ A_{\hat\mu\hat\nu\hat\nu\hat\mu} &= \rho(\hat\mu, \hat\mu) + \rho(\hat\nu, \hat\nu) - \mathrm{tr}(\rho) = -\,\mathrm{tr}(\rho_E), \end{aligned}$$

where $\mathbb{S}_F(\rho) = \rho_F - \mathrm{tr}(\rho_F) g_F$ is the trace-reversed part of ρ restricted to a facet $F \in \mathring{\mathscr{F}}$. Inserting into (2.15) yields

$$\frac{1}{4}\widetilde{\mathcal{R}\omega}(g \owedge J\rho) = \sum_{T\in\mathscr{T}} \int_T \langle G, \rho\rangle \omega_T + \sum_{F\in\mathring{\mathscr{F}}} \int_F \langle [\![\mathrm{I\!I}]\!], \mathbb{S}_F(\rho)\rangle \omega_F - \sum_{E\in\mathring{\mathscr{E}}} \int_E \Theta_E\, \rho_E\, \omega_E.$$

The right-hand side is exactly the *distributional densitized Einstein tensor* $\widetilde{G\omega}(\rho)$ defined and analyzed in [21, 22]. Note that $\langle [\![\mathrm{I\!I}]\!], \mathbb{S}_F(\rho)\rangle = \langle [\![\mathbb{S}_F \mathrm{I\!I}]\!], \rho_F\rangle = \langle [\![\overline{\mathrm{I\!I}}]\!], \rho_F\rangle$ with the trace-reversed second fundamental form $\overline{\mathrm{I\!I}}^{\hat\nu} = \mathrm{I\!I}^{\hat\nu} - H^{\hat\nu} g_F$. In general relativity the jump of the trace-reversed second fundamental form features in the well-known Israel junction condition [26].

3. Linearization of curvature

The test spaces $\mathcal{A}$ and $\mathring{\mathcal{A}}$ defined in (2.14), which provide test functions in the generalized curvature formula, depend on the metric tensor g, since the continuity properties are given in terms of the g-normal vector $\hat\nu$. While analyzing the convergence of the approximate Riemann curvature tensor as the metric g approaches the exact metric tensor $\bar g$, it is useful to work with g-independent test functions that remain unchanged as $g \to \bar g$. We will accomplish this by constructing a g-independent test space $\mathcal{U}$ in bijection with $\mathcal{A}$ in §3.1. This then allows us to quantify changes in quantities that depend on the metric. Specifically, introducing intermediate metrics between the exact metric $\bar g$ and its nonsmooth discretization g_h by $g(t) = \bar g + t(g_h - \bar g)$, we study how elements of $\mathcal{A}$ evolve as $g(t)$ evolves with a fictitious "time" variable t (in §3.2), followed by how each ingredient in the generalized Riemann curvature tensor (2.15) evolves as g evolves (in §3.3, §3.4, and §3.5).

3.1. **Metric-independent test space.** Let the symmetric dyadic product of two tensors a and b be denoted by $a \odot b = \frac{1}{2}(a \otimes b + b \otimes a)$. Consider the vector bundle that associates to each point p of Ω the vector space of linear combinations of symmetric dyadic products of $(N-2)$-forms. We denote its smooth sections by $\wedge^{N-2}(\mathscr{T})^{\odot 2} \equiv \wedge^{N-2}(\mathscr{T}) \odot \wedge^{N-2}(\mathscr{T})$. In other words, an element U of $\wedge^{N-2}(\mathscr{T})^{\odot 2}$ is a linear combination of tensors of the form $u \odot v$ for some piecewise smooth forms u and v in $\wedge^{N-2}(\mathscr{T})$. The restriction of U on an interior facet $F \in \mathring{\mathscr{F}}$, denoted by U_F, is a multi-valued function in general since its value depends on which of the elements sharing F is used for the restriction. Requiring it to be single-valued, as done next in (3.1), places inter-element continuity constraints. The following metric-independent test spaces with such continuity constraints on the parameter domain Ω, defined without using the Riemannian structure, are useful when perturbing the metric:

$$
\begin{aligned}
\mathcal{U} = \{U \in \textstyle\bigwedge^{N-2}(\mathscr{T})^{\odot 2} \,:\, & U|_F(X_1, \ldots, X_{N-2}, Y_1, \ldots, Y_{N-2}) \text{ is single-valued} \\
& \text{for any } X_i, Y_i \in \mathfrak{X}(F), F \in \mathring{\mathscr{F}}\}, \\
\mathring{\mathcal{U}} = \{U \in \mathcal{U} : U|_F(& X_1, \ldots, X_{N-2}, Y_1, \ldots, Y_{N-2}) = 0, \\
& \text{for any } X_i, Y_i \in \mathfrak{X}(F),\ F \in \mathscr{F}_\partial\}.
\end{aligned}
\tag{3.1}
$$

Remark 3.1. One can easily verify that continuity condition (3.1) implies that for all $X_i, Y_i \in \mathfrak{X}(E)$, $E \in \mathring{\mathscr{E}}$, $U|_E(X_1, \ldots, X_{N-2}, Y_1, \ldots, Y_{N-2})$ is single-valued.

We define a linear mapping $\mathbb{A} : \mathcal{U} \to \mathcal{T}^4_0(\mathscr{T})$ by

$$
(\mathbb{A}U)(X, Y, Z, W) = \langle U, \star(X^\flat \wedge Y^\flat) \odot \star(W^\flat \wedge Z^\flat)\rangle, \quad X, Y, Z, W \in \mathfrak{X}(\mathscr{T}), \tag{3.2}
$$

where $\star$ denotes the Hodge dual operator—see (A.3). Denote the range of $\mathbb{A}$ by

$$
\tilde{\mathcal{A}}_g \equiv \tilde{\mathcal{A}} := \{\mathbb{A}U \,:\, U \in \mathcal{U}\}. \tag{3.3}
$$

Although $\mathcal{U}$ is independent of g, note that $\mathcal{A}$, $\tilde{\mathcal{A}}$, and $\mathbb{A}$ all depend on g. When needed, we will emphasize this dependence by writing them as $\mathcal{A}_g$, $\tilde{\mathcal{A}}_g$, and $\mathbb{A}_g$, respectively. The next two lemmas show that (2.14) and (3.3) define the same spaces.

Lemma 3.2. *Let $g \in \mathrm{Reg}^+(\mathscr{T})$. Then $\tilde{\mathcal{A}}_g \subseteq \mathcal{A}_g$.*

Proof. Let $U \in \mathcal{U}$ and $A = \mathbb{A}_g U \in \tilde{\mathcal{A}}_g$. By definition (3.2), A clearly fulfills the symmetries in (2.12) of $\mathcal{A}_g$, so it suffices to prove that the continuity constraint (2.13) also hold.

Let $F \in \mathring{\mathscr{F}}$ and $T \in \mathscr{T}$ be an element containing F. Let $\{E_1, \ldots, E_{N-1}, E_N\}$ be any g-orthonormal frame with $E_1, \ldots, E_{N-1} \in \mathfrak{X}(F)$ and $E_N = \hat{\nu}_F^T$, abbreviated to $\hat{\nu}$ here. Note that $E_1, \ldots, E_{N-1}$ depend only on the tangential-tangential components of g restricted to F (as can be seen e.g. from Gram-Schmidt orthogonalization). Also, in the coordinates of the E_i frame, the matrix representation of g becomes the identity and $E_j^\flat = E_j$.

First consider the case $N > 2$. Then by (A.5), the Hodge duals

$$\star(E_1 \wedge E_N) = (-1)^{N-2} E_2 \wedge \cdots \wedge E_{N-1},$$
$$\star(E_2 \wedge E_N) = (-1)^{N-1} E_1 \wedge E_3 \wedge \cdots \wedge E_{N-1},$$

are expressed only in terms of tangential vectors on F. Letting $X = E_1$, $W = E_2$, we then see that

$$A(X, \hat{\nu}, \hat{\nu}, W) = \langle U, \star(E_1 \wedge E_N) \odot \star(E_2 \wedge E_N) \rangle$$

depends only on $U|_F(Z_1, \ldots, Z_{N-2}, Y_1, \ldots, Y_{N-2})$ where $Z_i, Y_i \in \{E_1, \ldots, E_{N-1}\}$ are tangential vectors in $\mathfrak{X}(F)$. Hence, by (3.1), $A(X, \hat{\nu}, \hat{\nu}, W)$ is single-valued on F. The same reasoning applies for all other choices of $X, W \in \{E_1, \ldots, E_{N-1}\}$ and hence for any $X, W \in \mathfrak{X}(F)$. Thus, we conclude that $A_{F\hat{\nu}\hat{\nu}F}$ is single-valued on all interior facets F.

In the $N = 2$ case, any $X, W \in \mathfrak{X}(F)$ must be a scalar multiple of E_1 and $\star(E_1 \wedge E_2) = 1$, so

$$A(E_1, \hat{\nu}, \hat{\nu}, E_1) = \langle U, \star(E_1 \wedge E_N) \odot \star(E_1 \wedge E_N) \rangle = U,$$

which is single-valued on interior facets F by (3.1). □

Next, we improve the inclusion of the previous lemma to an equality.

Lemma 3.3. *Let $g \in \mathrm{Reg}^+(\mathscr{T})$. Then $\tilde{\mathcal{A}}_g = \mathcal{A}_g$ and the mapping $\mathbb{A}_g : \mathcal{U} \to \mathcal{A}_g$ is a bijection.*

Proof. First, we prove that $\mathbb{A}_g$ is an injection. Let $U \in \mathcal{U}$ be such that for $X, Y, Z, W \in \mathfrak{X}(\mathscr{T})$,

$$\begin{aligned} 0 = (\mathbb{A}_g U)(X, Y, Z, W) &= \langle U, \star(X^\flat \wedge Y^\flat) \odot \star(W^\flat \wedge Z^\flat) \rangle \\ &= \langle U, \star(X^\flat \wedge Y^\flat) \otimes \star(W^\flat \wedge Z^\flat) \rangle. \end{aligned} \tag{3.4}$$

Within each mesh element, use a g-orthonormal basis $E^1, \ldots, E^N$ to expand U in components $U_{\alpha\beta} E^\alpha \odot E^\alpha$ for increasing multi-indices $\alpha = (\alpha_1, \ldots, \alpha_{N-2})$ and $\beta = (\beta_1, \ldots, \beta_{N-2})$, where E^α abbreviates $E^{\alpha_1} \wedge \cdots \wedge E^{\alpha_{N-2}}$. For each α and β, there are index pairs i, j and k, l and sign selections such that $\pm E^i \wedge E^j \wedge E^\alpha$ and

$\pm E^k \wedge E^l \wedge E^\beta$ equal the volume form ω. Then, using $X = \pm E^i$ and $W = \pm E^k$, with the selected signs, and $Y = E^j, Z = E^l$ in (3.4),

$$0 = U_{\alpha\beta}(\langle E^\alpha, \star(X^\flat \wedge Y^\flat)\rangle\langle E^\beta, \star(W^\flat \wedge Z^\flat)\rangle + \langle E^\beta, \star(X^\flat \wedge Y^\flat)\rangle\langle E^\alpha, \star(W^\flat \wedge Z^\flat)\rangle).$$

The term in parentheses equals 1 for $\alpha \neq \beta$ and equals 2 for $\alpha = \beta$, so $U_{\alpha\beta} = 0$. Repeating the argument for every coefficient of U, we conclude that $U = 0$.

It now suffices to prove that the dimensions of $\mathcal{A}_g$ and $\mathcal{U}$ are equal. The set of fourth order tensors fulfilling the symmetry conditions (2.12) is $N(N-1)\big(N(N-1)+2\big)/8$ dimensional. The number of constraints in (2.13) at a facet point is $N(N-1)/2$ (and the linear independence of the constraints can be verified by inserting a basis of $\mathfrak{X}(F)$). To see that these numbers match those for $\mathcal{U}$ in (3.1), first note that $\dim\big(\bigwedge^{N-2}(\mathscr{T})^{\odot 2}\big) = N(N-1)\big(N(N-1)+2\big)/8$. Next, the number of linearly independent constraints in (3.1) is also easily seen to be $N(N-1)/2$ on facets. Thus, the dimensions of $\mathcal{A}_g$ and $\mathcal{U}$ coincide and $\mathbb{A}_g$ is a bijection. ☐

Recall that (2.15) generalized the densitized Riemann curvature to a functional $\widetilde{\mathcal{R}\omega}(A)$ acting on A in $\mathcal{A}$. In view of the bijection $\mathbb{A}$, we now consider the corresponding functional on U, defined by

$$\widetilde{\mathcal{Q}\omega}(U) := \widetilde{\mathcal{R}\omega}(\mathbb{A}U). \tag{3.5}$$

Theorem 3.4. *The generalized curvature functional $\widetilde{\mathcal{Q}\omega}(U)$ can be calculated using metric-independent test functions U by*

$$\widetilde{\mathcal{Q}\omega}(U) = \sum_{T\in\mathscr{T}} \int_T \langle \mathcal{R}, \mathbb{A}U\rangle \omega_T + 4 \sum_{F\in\mathring{\mathscr{F}}} \int_F \langle [\![\mathrm{II}]\!], (\mathbb{A}U)_{\cdot\hat{\nu}\hat{\nu}\cdot}\rangle \omega_F + 4 \sum_{E\in\mathring{\mathscr{E}}} \int_E \Theta_E\, (\mathbb{A}U)_{\hat{\mu}\hat{\nu}\hat{\nu}\hat{\mu}}\, \omega_E$$

for all $U \in \mathring{\mathcal{U}}$.

Proof. Apply Lemma 3.3. ☐

Remark 3.5 (Alternative representation of $\mathbb{A}$). Extend the Hodge star to the symmetric product of k-forms, denote it by $\star^{\odot 2} : \bigwedge^k(\mathscr{T})^{\odot 2} \to \bigwedge^{N-k}(\mathscr{T})^{\odot 2}$, and define it by

$$\star^{\odot 2}(u \odot v) = (\star u) \odot (\star v), \qquad u, v \in \textstyle\bigwedge^k(\mathscr{T}).$$

We claim that an alternative representation of $\mathbb{A} : \mathcal{U} \to \mathcal{A}$ in (3.2) is

$$\mathbb{A} = -\star^{\odot 2}. \tag{3.6}$$

Indeed, given $U \in \mathcal{U}$, (3.2) can be rewritten as

$$\begin{aligned}(\mathbb{A}U)(X,Y,Z,W) &= \langle U, \star(X^\flat \wedge Y^\flat) \odot \star(W^\flat \wedge Z^\flat)\rangle \\ &= \frac{1}{4}\langle \star^{\odot 2}U, (X^\flat \wedge Y^\flat) \odot (W^\flat \wedge Z^\flat)\rangle \\ &= \langle \star^{\odot 2}U, X^\flat \otimes Y^\flat \otimes W^\flat \otimes Z^\flat\rangle = -(\star^{\odot 2}U)(X,Y,Z,W),\end{aligned}$$

where we have used (A.11) and (A.2).

Remark 3.6 (The Uhlenbeck trick). Our technique of putting the metric-dependent test function space $\mathcal{A}$ in isomorphism with the fixed, metric-independent space $\mathcal{U}$ may be viewed as a variant of a well-known idea of Karen Uhlenbeck: cf. [24] or [40, Section 9.4].

Remark 3.7 (Finite elements for $\mathcal{U}$)*.* In 2D, (2.15) reduces to (2.19), and the test space $\mathcal{U}$ can be discretized by Lagrange finite elements. In three spatial dimensions $\mathcal{U}$ consists of Regge functions, which are *tt*-continuous, and can be discretized using Regge finite elements. The specialization of (2.15) in 3D is presented and discussed in §6.2.

Remark 3.8 (Curvature operator)*.* Often, the smooth Riemann curvature tensor $\mathcal{R}$ is identified with an operator acting on bivectors (see e.g., [33, Section 3.1.2]) due to the symmetries of $\mathcal{R}$. In our context, we can similarly define, within each element T a *curvature operator* acting on 2-forms by setting its quadratic form, $\mathfrak{Q} : \wedge^2(T) \times \wedge^2(T) \to \mathbb{R}$ by

$$\mathfrak{Q}(X^\flat \wedge Y^\flat, W^\flat \wedge Z^\flat) := \mathcal{R}(X, Y, Z, W), \qquad \text{for all } X, Y, Z, W \in \mathfrak{X}(T). \tag{3.7}$$

Clearly, $\mathfrak{Q}$ is a symmetric bilinear form in its two arguments, and generates a selfadjoint curvature operator. Next, define

$$\mathcal{Q} = \mathbb{A}^{-1}\mathcal{R}. \tag{3.8}$$

Then, due to (3.2), the equation $\mathbb{A}\mathcal{Q} = \mathcal{R}$ can be written as

$$\langle \mathcal{Q}, \star(X^\flat \wedge Y^\flat) \odot \star(W^\flat \wedge Z^\flat)\rangle = \mathcal{R}(X, Y, Z, W)$$

for all $X, Y, Z, W \in \mathfrak{X}(T)$. Comparing with (3.7), we see that $\mathcal{Q}$ and $\mathfrak{Q}$ are related by

$$\mathfrak{Q}(X^\flat \wedge Y^\flat, W^\flat \wedge Z^\flat) = \langle \mathcal{Q}, \star(X^\flat \wedge Y^\flat) \odot \star(W^\flat \wedge Z^\flat)\rangle = \mathcal{R}(X, Y, Z, W).$$

Thus $\mathcal{Q}, \mathfrak{Q}$, and $\mathcal{R}$ all contain the same information. A generalized densitized version of $\mathcal{Q}$ is the functional $\widetilde{\mathcal{Q}\omega}$ of Theorem 3.4. In §6.2, we will revisit the curvature operator in 3D.

We conclude this subsection by giving coordinate formulas for the map $\mathbb{A}$ and its inverse, anticipating its utility in computations. Let $\hat{\varepsilon}^{i_1\cdots i_N} = \varepsilon^{i_1\cdots i_N}/\sqrt{\det g}$, with $\varepsilon^{i_1\cdots i_N} = \varepsilon_{i_1\ldots i_N}$ denoting the standard permutation symbol, whose value is 1, -1, or 0, when $(i_1, \ldots, i_N)$ is an even, odd, or not a permutation of $(1, \ldots, N)$, respectively. In lowered indices it reads $\hat{\varepsilon}_{i_1\ldots i_N} = \sqrt{\det g}\, \varepsilon_{i_1\ldots i_N}$. Let $\alpha = (\alpha_1, \ldots, \alpha_{N-2})$ and $\beta = (\beta_1, \ldots, \beta_{N-2})$ denote multi-indices of integers $\alpha_m, \beta_m \in \{1, \ldots, N\}$. Let $g^{\alpha\beta} = g^{\alpha_1\beta_1} \ldots g^{\alpha_{N-2}\beta_{N-2}}$. Mixing integer indices p, q and multiindex α, we write $\hat{\varepsilon}^{pq\alpha}$ for $\hat{\varepsilon}^{p,q,\alpha_1\cdots\alpha_{N-2}}$ (and similarly $\varepsilon^{pq\alpha}$). Given $U \in \mathcal{U}$, and a coordinate frame $\{\partial_i\}_{i=1}^N$, we wish to relate the coefficients

$$U_{\alpha\beta} \equiv U_{\alpha_1,\ldots,\alpha_{N-2},\beta_1,\ldots,\beta_{N-2}} := U(\partial_{\alpha_1}, \ldots, \partial_{\alpha_{N-2}}, \partial_{\beta_1}, \ldots, \partial_{\beta_{N-2}}),$$

with $A_{ijkl} = A(\partial_i, \partial_j, \partial_k, \partial_l)$ when $A = \mathbb{A}U$. We extend the summation convention to multi-indices α and β (so a sum over all components α_m and β_m is implied in the formula (3.9) below).

Proposition 3.9. *For any $U \in \mathcal{U}$, using the above-mentioned coordinate notation,*

$$\begin{aligned} [\mathbb{A}U]_{ijkl} &= \frac{-1}{[(N-2)!]^2} \hat{\varepsilon}^{pq\alpha}\, \hat{\varepsilon}^{rs\beta}\, U_{\alpha\beta}\, g_{pi} g_{qj} g_{rk} g_{sl}, \\ [\mathbb{A}U]^{ijkl} &= \frac{-1}{[(N-2)!]^2} \hat{\varepsilon}^{ij\alpha}\, \hat{\varepsilon}^{kl\beta}\, U_{\alpha\beta}. \end{aligned} \tag{3.9}$$

Proof. By the definition of $\mathbb{A}$ in (3.2), $[\mathbb{A}U]_{ijkl} = \langle U, \star(\partial_i^\flat \wedge \partial_j^\flat) \odot \star(\partial_l^\flat \wedge \partial_k^\flat)\rangle$. Noting that (A.7) implies

$$\star(\partial_i^\flat \wedge \partial_j^\flat) = \frac{\sqrt{\det g}}{(N-2)!}\,\varepsilon_{ij\,\beta}\,dx^{\beta_1} \wedge \cdots \wedge dx^{\beta_{N-2}},$$

we obtain, using (A.9),

$$[\mathbb{A}U]_{ijkl} = \frac{\det g}{[(N-2)!]^2} U_{\alpha\beta} g^{\alpha\gamma} g^{\beta\mu} \varepsilon_{ij\,\alpha} \varepsilon_{lk\,\mu}.$$

Now simplifying using $\varepsilon_{ij\,\alpha} g^{\alpha\gamma} = g_{pi} g_{qj} \varepsilon_{rs\,\alpha} g^{rp} g^{sq} g^{\alpha\gamma} = g_{pi} g_{qj} \varepsilon^{pq\,\alpha} \det(g^{-1})$, we obtain the first expression in (3.9). The second follows by the direct computation of $[\mathbb{A}U]^{ijkl} = g^{ip} g^{jq} g^{kr} g^{ls} [\mathbb{A}U]_{pqrs}$. □

Proposition 3.10. *The inverse $\mathbb{A}^{-1} : \mathcal{A} \to \mathcal{U}$ has the explicit form*

$$\begin{aligned}(\mathbb{A}^{-1}A)&(X_1, \ldots, X_{N-2}, Y_1, \ldots, Y_{N-2}) \\ &= -\frac{1}{4}\langle A, \star(X_1^\flat \wedge \cdots \wedge X_{N-2}^\flat) \odot \star(Y_1^\flat \wedge \cdots \wedge Y_{N-2}^\flat)\rangle,\end{aligned} \tag{3.10}$$

for any $A \in \mathcal{A}$, and has the representation $\mathbb{A}^{-1} = -\star^{\odot 2}$. In coordinates, for multi-indices $\alpha = (\alpha_1, \ldots, \alpha_{N-2})$ and $\beta = (\beta_1, \ldots, \beta_{N-2})$,

$$(\mathbb{A}^{-1}A)_{\alpha\beta} = -\frac{1}{4}\hat{\varepsilon}_{\alpha\,ij}\hat{\varepsilon}_{\beta\,kl} A^{ijkl}. \tag{3.11}$$

Proof. To show that (3.10) is indeed the inverse we prove that $\mathbb{A}^{-1}\mathbb{A} = \mathrm{id}$. First, we prove the representation $\mathbb{A}^{-1} = -\star^{\odot 2}$ of (3.10). By (A.11),

$$\begin{aligned}(\mathbb{A}^{-1}A)&(X_1, \ldots, X_{N-2}, Y_1, \ldots, Y_{N-2}) \\ &= -\frac{1}{4}\langle A, \star(X_1^\flat \wedge \cdots \wedge X_{N-2}^\flat) \odot \star(Y_1^\flat \wedge \cdots \wedge Y_{N-2}^\flat)\rangle \\ &= -\frac{1}{((N-2)!)^2}\langle \star^{\odot 2}A, X_1^\flat \wedge \cdots \wedge X_{N-2}^\flat \odot Y_1^\flat \wedge \cdots \wedge Y_{N-2}^\flat\rangle \\ &= -\langle \star^{\odot 2}A, X_1^\flat \otimes \cdots \otimes X_{N-2}^\flat \otimes Y_1^\flat \otimes \cdots \otimes Y_{N-2}^\flat\rangle \\ &= -(\star^{\odot 2}A)(X_1, \ldots, X_{N-2}, Y_1, \ldots, Y_{N-2}).\end{aligned}$$

Then, we have for all $U \in \mathcal{U}$ with (3.6) (and that $\star \circ \star = \pm 1$)

$$\mathbb{A}^{-1}(\mathbb{A}U) = \star^{\odot 2}(\star^{\odot 2}U) = U.$$

Coordinate expression (3.11) can be derived analogously to (3.9). □

3.2. **Evolution of test functions.** In the remainder of this section, we consider a family of metrics $g(t) \in C^1(\mathbb{R}, \mathcal{S}^+)$ parameterized by time t and examine how relevant quantities change as $g(t)$ changes with time t. Let $\dot{g}(t) = (dg_{ij}/dt)dx^i \otimes dx^j$ be in a t-independent coordinate coframe dx^i. The specific task for this subsection is an investigation of how the test function $A \in \mathcal{A}$ changes as the metric g evolves. Here, we rely on definition (3.3) and mapping (3.2).

To this end, we need some preliminaries. Let $X_i \in \mathfrak{X}(\mathscr{T})$ be time-independent vector fields and let $\sigma \in \mathcal{T}_0^2(\mathscr{T})$. Let $L_\sigma : \mathfrak{X}(\mathscr{T}) \to \mathfrak{X}(\mathscr{T})$ be the endomorphism defined by

$$g(L_\sigma X_1, X_2) = \sigma(X_1, X_2). \tag{3.12}$$

By trace of σ we mean the usual trace of the linear operator L_σ,

$$\operatorname{tr}(\sigma) := \operatorname{tr}(L_\sigma) = g^{ij}\sigma_{ij}. \tag{3.13}$$

It is easy to prove (e.g., using the Jacobi formula for determinant derivative) that the derivative of the time-dependent Riemannian volume form $\omega \equiv \omega_{g(t)}$ is given by

$$\frac{d}{dt}\omega_{g(t)}(X_1,\dots,X_N) = \frac{1}{2}\operatorname{tr}(\dot g)\,\omega_{g(t)}(X_1,\dots,X_N), \tag{3.14}$$

where the $\operatorname{tr}(\dot g)$ is calculated as in (3.13). Given any $A \in \mathcal{T}^k_0(\mathscr{T})$, define $L^{(\ell)}_\sigma A \in \mathcal{T}^k_0(\mathcal{T})$, for any $\ell = 1,\dots,k$, by

$$(L^{(\ell)}_\sigma A)(X_1,\dots,X_k) := A(X_1,\dots,X_{\ell-1}, L_\sigma X_\ell, X_{\ell+1},\dots,X_k). \tag{3.15}$$

When σ is symmetric, it is easy to verify that

$$\langle L^{(\ell)}_\sigma A, B\rangle = \langle A, L^{(\ell)}_\sigma B\rangle, \qquad A,B \in \mathcal{T}^k_0(\mathscr{T}), \tag{3.16}$$

$$\langle L^{(\ell)}_\sigma A, B\rangle = \langle L^{(1)}_\sigma A, B\rangle, \qquad A,B \in \mathcal{A},\ \ell = 1,\dots,4. \tag{3.17}$$

Define $\dot A \in \mathcal{T}^k_0(\mathscr{T})$ by

$$\dot A(Y_1,\dots,Y_k) = \frac{d}{dt}A(Y_1,\dots,Y_k) - \sum_{i=1}^k A\left(Y_1,\dots,\frac{dY_i}{dt},\dots,Y_k\right)$$

for possibly time-dependent vector fields $Y_i \in \mathfrak{X}(\mathscr{T})$. When Y_i are time-independent, the last sum vanishes. In components of the time-independent coordinate frame $\{\partial_1,\dots,\partial_N\}$, this means that the components of $\dot A$ are obtained by differentiating the components of A, i.e., $\dot A_{i_1,\dots,i_k} = dA_{i_1,\dots,i_k}/dt$. Using this notation, we have the following lemma.

Lemma 3.11. *For any $A, B \in \mathcal{T}^k_0(\mathscr{T})$,*

$$\frac{d}{dt}\langle A,B\rangle = \langle \dot A, B\rangle + \langle A, \dot B\rangle - \sum_{\ell=1}^k \langle L^{(\ell)}_{\dot g} A, B\rangle.$$

Proof. Since $\dot g^{ij} = -g^{ik}\,\dot g_{kl}\,g^{lj}$, by the product rule on (A.9),

$$\begin{aligned}\frac{d}{dt}\langle A,B\rangle &= \frac{d}{dt}(A_{i_1\dots i_k} g^{i_1 j_1}\dots g^{i_k j_k} B_{j_1\dots j_k})\\ &= \langle \dot A, B\rangle + \langle A, \dot B\rangle - \sum_{\ell=1}^k A_{i_1\dots i_k}\, g^{i_1 j_1}\cdots(g^{i_\ell k}\dot g_{kl} g^{l j_\ell})\cdots g^{i_k j_k}\, B_{j_1\dots j_k}\end{aligned}$$

and the ℓth summand in the last sum is easily seen to equal $\langle L^{(\ell)}_{\dot g} A, B\rangle$. □

Lemma 3.12. *Let $\omega \in \mathcal{T}^N_0(\mathscr{T})$ be the volume form. Then for symmetric 2-tensors $\sigma \in \mathcal{S}(\mathscr{T})$,*

$$\sum_{i=1}^N L^{(i)}_\sigma \omega = \operatorname{tr}(\sigma)\,\omega. \tag{3.18}$$

Proof. Let $X_i \in \mathfrak{X}(\mathscr{T})$, and $E_i \in \mathfrak{X}(\mathscr{T})$ be an oriented g-orthonormal frame. Further, let $b_i \in \mathbb{R}^N$ denote the column vector whose jth component equals $g(X_j, E_i)$, and L be the matrix whose (i,j)th entry is $\sigma(E_i,E_j)$. Then, by (3.12), the jth component of Lb_i equals $g(L_\sigma X_j, E_i)$ and the trace of the matrix L equals $\operatorname{tr}(\sigma)$.

Combined with the fact that $\omega(X_1,\dots,X_N) = \det[b_1,\dots,b_N]$, we see that (3.18) is the same as

$$\sum_{i=1}^{N} \det[b_1,\dots,L\,b_i,\dots,b_N] = \operatorname{tr}(L)\,\det[b_1,\dots,b_N]. \tag{3.19}$$

The identity (3.19) actually holds for arbitrary matrices $L \in \mathbb{R}^{N\times N}$ and vectors $b_i \in \mathbb{R}^N$ and can be readily verified as follows. Both sides of (3.19) are alternating and linear in b_i, so they are N-forms, and can only differ by a scalar factor. That the scalar factor, in this occasion, is $\operatorname{tr}(L)$, can be seen by substituting the standard Euclidean unit basis vectors for b_i. □

Lemma 3.13. *For time-independent vector fields $X, Y, Z, W \in \mathfrak{X}(\mathscr{T})$ and $A \in \mathcal{A}$, the time-derivative of $A(X,Y,Z,W)$ is given by*

$$\begin{aligned}\dot{A}(X,Y,Z,W) &= A(L_{\dot g}X,Y,Z,W) + A(X,L_{\dot g}Y,Z,W)\\ &\quad + A(X,Y,L_{\dot g}Z,W) + A(X,Y,Z,L_{\dot g}W) - \operatorname{tr}(\dot g)A(X,Y,Z,W).\end{aligned}$$

Proof. By Lemma 3.3, there exists a time-independent $U \in \mathcal{U}$ such that $A = \mathbb{A}_g U$, which by (3.2) is the same as $A(X,Y,Z,W) = \langle U, V\rangle$ with $V = \star(X^\flat \wedge Y^\flat) \odot \star(W^\flat \wedge Z^\flat)$. Since $\dot U = 0$, by Lemma 3.11,

$$\dot{A}(X,Y,Z,W) = \langle U, \dot V\rangle - \sum_{i=1}^{2(N-2)} \langle L_{\dot g}^{(i)} U, V\rangle.$$

We compute $\dot V$ using

$$\begin{aligned}\frac{d}{dt}\big(\star(X^\flat \wedge Y^\flat)\big) &= \frac{d}{dt}\omega(X,Y,\dots) && \text{by (A.8)},\\ &= \frac{1}{2}\operatorname{tr}(\dot g)\ \star(X^\flat \wedge Y^\flat) && \text{by (3.14)}.\end{aligned}$$

Then

$$\dot{A}(X,Y,Z,W) = \operatorname{tr}(\dot g)\langle U, V\rangle - \sum_{i=1}^{2(N-2)} \langle U, L_{\dot g}^{(i)} V\rangle. \tag{3.20}$$

By Lemma 3.12 and (A.8),

$$\begin{aligned}2\operatorname{tr}(\dot g)V &= \star((L_{\dot g}X)^\flat \wedge Y^\flat) \odot \star(W^\flat \wedge Z^\flat) + \star(X^\flat \wedge (L_{\dot g}Y)^\flat) \odot \star(W^\flat \wedge Z^\flat)\\ &\quad + \star(X^\flat \wedge Y^\flat) \odot \star((L_{\dot g}W)^\flat \wedge Z^\flat) + \star(X^\flat \wedge Y^\flat) \odot \star(W^\flat \wedge (L_{\dot g}Z)^\flat) + \sum_{i=1}^{2(N-2)} L_{\dot g}^{(i)} V.\end{aligned}$$

Taking the inner product with U on both sides and adding to (3.20), we obtain the result. □

3.3. **Evolution of Riemann curvature tensor.** We proceed to study how each term in our formula for the generalization of Riemann curvature (2.15) evolves in time with $g(t)$. This subsection focuses on the integrand over T in (2.15) and gives its time derivative (in Lemma 3.15).

Given a k-permutation $\pi \in \mathfrak{S}_k$, let $(S_\pi A)(X_1,\dots,X_k) = A(X_{\pi(1)},\dots,X_{\pi(k)})$ for all $X_i \in \mathfrak{X}(\mathscr{T})$ and $A \in \mathfrak{T}^k_0(\mathscr{T})$. A transposition (i,j) thus generates $S_{(i,j)}$ which swaps the ith and jth input arguments of a tensor. The transposition $S_{(2,3)}$ on 4-tensors is of particular interest to us: we abbreviate it to S, i.e.,

$$(SA)(X, Y, Z, W) = A(X, Z, Y, W), \tag{3.21}$$

for any $X, Y, Z, W \in \mathfrak{X}(\mathscr{T})$ and $A \in \mathfrak{T}^4_0(\mathscr{T})$, or, in coordinates, $S^{pqrs}_{ijkl} = \delta^p_i \delta^r_j \delta^q_k \delta^s_l$ so that $S^{pqrs}_{ijkl} A_{pqrs} = A_{ikjl}$. It is easy to see that $S_{(i,j)}$ is self-adjoint,

$$\langle S_{(i,j)} A, B\rangle = \langle A, S_{(i,j)} B\rangle \tag{3.22}$$

for all $A, B \in \mathfrak{T}^4_0(\mathscr{T})$. The operation of skew-symmetrizing a tensor A with respect to its ith and jth arguments is $P_{ij} = \frac{1}{2}(I - S_{(i,j)})$. Of particular interest to us is $P = P_{13} \circ P_{24} : \mathfrak{T}^4_0(\mathscr{T}) \to \mathfrak{T}^4_0(\mathscr{T})$, which skew-symmetrizes with respect to arguments 2, 4 followed by 1, 3. Expanding P in terms of $S_{(i,j)}$,

$$P = \frac{1}{4}\Big(I - S_{(1,3)} - S_{(2,4)} + S_{(1,3)} \circ S_{(2,4)}\Big). \tag{3.23}$$

Disjoint transpositions commute, so $S_{(1,3)} \circ S_{(2,4)} = S_{(2,4)} \circ S_{(1,3)}$. This fact, together with (3.22), immediately implies that P is a self-adjoint projection,

$$\langle PA, B\rangle = \langle A, PB\rangle, \qquad P^2 = P. \tag{3.24}$$

Note that for any covariant 4-tensor A, combining (3.21) and (3.23),

$$\begin{aligned}(SPA)(X, Y, Z, W) &= (PA)(X, Z, Y, W) \\ &= \frac{1}{4}\Big[A(X, Z, Y, W) - A(Y, Z, X, W) + A(Y, W, X, Z) - A(X, W, Y, Z)\Big].\end{aligned} \tag{3.25}$$

Also note that by the symmetries of the test space $\mathcal{A}$,

$$(PSA)(X, Y, Z, W) = (SA)(X, Y, Z, W), \qquad \text{for any } A \in \mathcal{A}. \tag{3.26}$$

Using these operators, as well as the second order covariant derivative (see (A.1)), we obtain the following characterization of the derivative of $\mathcal{R}$.

Lemma 3.14. *The action of the time-derivative of the Riemann curvature tensor* (2.6) *on time-independent vector fields $X, Y, Z, W \in \mathfrak{X}(T)$, for any $T \in \mathscr{T}$, is given by*

$$\dot{\mathcal{R}}(X, Y, Z, W) = 2(SP\nabla^2 \dot{g})(X, Y, Z, W) + \frac{1}{2}\Big[\mathcal{R}(X, Y, Z, L_{\dot g} W) + \mathcal{R}(X, Y, L_{\dot g} Z, W)\Big].$$

Proof. By [40, Proposition 2.3.5] (accounting for the different sign convention of the curvature endomorphism used there),

$$\begin{aligned}\dot{\mathcal{R}}(X, Y, Z, W) = \frac{1}{2}\Big[&(\nabla^2_{X,Z}\dot g)(Y, W) - (\nabla^2_{X,W}\dot g)(Y, Z) + \dot g(R_{X,Y}Z, W) \\ &+ (\nabla^2_{Y,W}\dot g)(X, Z) - (\nabla^2_{Y,Z}\dot g)(X, W) - \dot g(R_{X,Y}W, Z)\Big].\end{aligned}$$

(3.12) and (2.6) imply $\dot g(R_{X,Y}Z, W) = g(R_{X,Y}Z, L_{\dot g}W) = \mathcal{R}(X, Y, Z, L_{\dot g}W)$. Hence the result follows from (3.25). □

Lemma 3.15. *Let $A \in \mathring{\mathcal{A}}$, $T \in \mathscr{T}$. Then*

$$\frac{d}{dt}\Big(\langle \mathcal{R}, A\rangle \omega_T\Big) = \Big(2\langle \nabla^2 \dot g, SA\rangle + \langle L^{(1)}_{\dot g}\mathcal{R}, A\rangle - \frac{\mathrm{tr}(\dot g)}{2}\langle \mathcal{R}, A\rangle\Big)\,\omega_T.$$

Proof. By Lemma 3.14 and (3.14),

$$\frac{d}{dt}\Big(\langle \mathcal{R}, A\rangle \omega_T\Big) = \big(\langle \dot{\mathcal{R}}, A\rangle + \langle \mathcal{R}, \dot{A}\rangle - \sum_{\ell=1}^{4}\langle L_{\dot g}^{(\ell)}\mathcal{R}, A\rangle + \frac{1}{2}\langle\, \mathrm{tr}(\dot g)\mathcal{R}, A\rangle\big)\omega_T.$$

The first two terms admit the following identities:

$$\begin{aligned}
\langle \mathcal{R}, \dot{A}\rangle &= \sum_{\ell=1}^{4}\langle L_{\dot g}^{(\ell)}\mathcal{R}, A\rangle - \langle\, \mathrm{tr}(\dot g)\mathcal{R}, A\rangle, && \text{by Lemma 3.13 and (3.16)},\\
\langle \dot{\mathcal{R}}, A\rangle &= 2\langle SP\nabla^2\dot g, A\rangle + \frac{1}{2}\sum_{\ell=3}^{4}\langle L_{\dot g}^{(\ell)}\mathcal{R}, A\rangle, && \text{by Lemma 3.14},\\
&= 2\langle \nabla^2\dot g, SA\rangle + \langle L_{\dot g}^{(1)}\mathcal{R}, A\rangle, && \text{by (3.22), (3.24), (3.26), and (3.17).}
\end{aligned}$$

Using them and simplifying, we obtain the result. □

3.4. **Evolution of second fundamental form.** Next, we consider the codimension 1 boundary term in (2.15) and compute its time derivative (in Lemma 3.18). Let us begin by recalling that the time derivative of the Levi-Civita connection can be computed by differentiating the terms in the Koszul formula: specifically, by [40, Proposition 2.3.1],

$$g\left(\frac{d}{dt}(\nabla_Y X), Z\right) = \frac{1}{2}\big[(\nabla_X \dot g)(Y,Z) + (\nabla_Y \dot g)(X,Z) - (\nabla_Z \dot g)(X,Y)\big] \tag{3.27}$$

for any time-independent vector fields $X, Y, Z \in \mathfrak{X}(\mathscr{T})$. Since both sides are linear in Z, (3.27) also holds for time-dependent $Z \equiv Z(t) \in \mathfrak{X}(\mathscr{T})$ by expanding $Z(t)$ in a time-independent frame such as ∂_i. The next auxiliary result we need is as follows (cf. [21, Lemma 2.5]).

Lemma 3.16. *Let $F \in \mathscr{F}$, $X(t) \in \mathfrak{X}(\mathscr{T})$ be a possibly time-dependent vector field, and let $\hat\nu$ denote a g-normal vector on F. Then, abbreviating $\dot g(\hat\nu,\hat\nu)$ to $\dot g_{\hat\nu\hat\nu}$,*

$$\frac{d\hat\nu}{dt} = \frac{1}{2}\dot g_{\hat\nu\hat\nu}\hat\nu - L_{\dot g}\hat\nu, \tag{3.28}$$

$$\frac{d}{dt}g(\hat\nu, X) = \frac{1}{2}\dot g_{\hat\nu\hat\nu} g(\hat\nu, X) + g(\hat\nu, \dot X). \tag{3.29}$$

Proof. Let $\{\hat\tau_i\}_{i=1}^{N-1}$ denote a $g(t)$-orthonormal basis of the tangent space at a point in F. Since the tangent space $\mathfrak{X}(F)$ depends only on the triangulation $\mathscr{T}$ (and is independent of t), the difference quotients $(\hat\tau_i(t_2) - \hat\tau_i(t_1))/(t_2 - t_1)$ are in $\mathfrak{X}(F)$ at different times t_1, t_2, so $\dot{\hat\tau}_i = d\hat\tau_i/dt \in \mathfrak{X}(F)$ and $g(\hat\nu, \dot{\hat\tau}_i) = 0$. Hence, by the product rule,

$$0 = \frac{d}{dt}g(\hat\nu, \hat\tau_i) = \dot g(\hat\nu, \hat\tau_i) + g(\dot{\hat\nu}, \hat\tau_i).$$

Since $g(\hat\nu,\hat\nu) = 1$ for all t, we also have

$$0 = \frac{d}{dt}g(\hat\nu,\hat\nu) = \dot g(\hat\nu,\hat\nu) + 2g(\dot{\hat\nu},\hat\nu).$$

After differentiating the expansion of $\dot{\hat\nu}$ in the basis $\{\hat\tau_1, \ldots, \hat\tau_{N-1}, \hat\nu\}$ these identities imply,

$$\frac{d\hat\nu}{dt} = g(\dot{\hat\nu}, \hat\nu)\hat\nu + \sum_{i=1}^{N-1} g(\dot{\hat\nu}, \hat\tau_i)\hat\tau_i = -\frac{1}{2}\dot g(\hat\nu,\hat\nu)\hat\nu - \sum_{i=1}^{N-1} \dot g(\hat\nu, \hat\tau_i)\hat\tau_i = \frac{1}{2}\dot g(\hat\nu,\hat\nu)\hat\nu - L_{\dot g}\hat\nu,$$

where we have used (3.12) in the last step. This proves (3.28). Using it, (3.29) readily follows by applying the product rule and simplifying. □

Let $T \in \mathscr{T}$ and $F \in \triangle_{-1}T$ be a facet of T. Recall that per the notation in (2.3), for facet tangent vectors $X, Y, Z \in \mathfrak{X}(F)$,

$$(\nabla\sigma)_{F\hat\nu F}(X,Y) := (\nabla\sigma)(X,\hat\nu,Y), \qquad (\nabla\sigma)_{\hat\nu FF}(X,Y) := (\nabla\sigma)(\hat\nu, X, Y), \tag{3.30a}$$

$$(\nabla\sigma)_F(X,Y,Z) := (\nabla\sigma)(X,Y,Z), \qquad \sigma_F(X,Y) := \sigma(X,Y). \tag{3.30b}$$

We also use the interior product $\lrcorner$ (see (A.15)) and a triple product notation

$$\tau : \sigma : \eta = \tau_{ij}\sigma^{jk}\eta_k{}^i \text{ for } \tau, \sigma, \eta \in \mathcal{T}^2_0(\mathscr{T}) \tag{3.31}$$

with the usual metric-based index raising. The following lemma is needed for the main result (Theorem 3.20) of this section.

Lemma 3.17. *Let $T \in \mathscr{T}$, $F \in \triangle_{-1}T$ be a facet of T, $A \in \mathcal{A}$, and $\sigma \in \mathcal{T}^2_0(T)$. Then*

$$\langle \nabla(\hat\nu \lrcorner \sigma), A_{F\hat\nu\hat\nu F}\rangle = \langle (\nabla\sigma)_{F\hat\nu F}, A_{F\hat\nu\hat\nu F}\rangle - \mathrm{I\!I}^{\hat\nu} : \sigma_F : A_{F\hat\nu\hat\nu F}.$$

Proof. Let $X, Y \in \mathfrak{X}(F)$. It is easy to see that $g(\nabla_X\hat\nu, \hat\nu) = 0$. Hence we may expand $\nabla_X\hat\nu$ in a tangential g-orthonormal frame $\{\hat\tau_i\}_{i=1}^{N-1}$ of $\mathfrak{X}(F)$ as $\nabla_X\hat\nu = \sum_{i=1}^{N-1} g(\nabla_X\hat\nu, \hat\tau_i)\hat\tau_i$. Using it,

$$\begin{aligned}(\nabla\sigma)_{F\hat\nu F}(X,Y) &= (\nabla_X\sigma)(\hat\nu, Y) = \nabla_X(\sigma(\hat\nu,Y)) - \sigma(\hat\nu, \nabla_XY) - \sigma(\nabla_X\hat\nu, Y)\\ &= \nabla_X(\sigma(\hat\nu,Y)) - \sigma(\hat\nu,\nabla_XY) - \sum_{i=1}^{N-1} g(\nabla_X\hat\nu, \hat\tau_i)\sigma(\hat\tau_i, Y)\\ &= (\nabla_X(\hat\nu\lrcorner\sigma))(Y) + \sum_{i=1}^{N-1} \mathrm{I\!I}^{\hat\nu}(X,\hat\tau_i)\sigma(\hat\tau_i, Y).\end{aligned}$$

Hence

$$\langle (\nabla\sigma)_{F\hat\nu F}, A_{F\hat\nu\hat\nu F}\rangle = \langle \nabla(\hat\nu\lrcorner\sigma), A_{F\hat\nu\hat\nu F}\rangle + \sum_{i,j,k=1}^{N-1} \mathrm{I\!I}^{\hat\nu}(\hat\tau_i,\hat\tau_k)\sigma(\hat\tau_k,\hat\tau_j)A(\hat\tau_i,\hat\nu,\hat\nu,\hat\tau_j)$$

and the lemma follows since

$$A(\hat\tau_i, \hat\nu, \hat\nu, \hat\tau_j) = A(\hat\tau_j, \hat\nu, \hat\nu, \hat\tau_i) \tag{3.32}$$

by the symmetries of A in (2.12). □

Lemma 3.18. *Let $T \in \mathscr{T}$ and $F \in \triangle_{-1}T$ be a facet of T. Then, for any $A \in \mathring{\mathcal{A}}$*

$$\frac{d}{dt}\Big(\langle \mathrm{I\!I}^{\hat\nu}, A_{F\hat\nu\hat\nu F}\rangle\omega_F\Big) = \frac{1}{2}\Big\langle (\dot g_{\hat\nu\hat\nu} - \operatorname{tr}\dot g_F)\mathrm{I\!I}^{\hat\nu} + 2(\nabla\dot g)_{F\hat\nu F} - (\nabla\dot g)_{\hat\nu FF},\ A_{F\hat\nu\hat\nu F}\Big\rangle\omega_F.$$

Proof. Since Lemma 3.11 gives

$$\frac{d}{dt}\langle \mathrm{I\!I}^{\hat\nu}, A_{F\hat\nu\hat\nu F}\rangle = \langle \dot{\mathrm{I\!I}}^{\hat\nu}, A_{F\hat\nu\hat\nu F}\rangle + \langle \mathrm{I\!I}^{\hat\nu}, \frac{d}{dt}A_{F\hat\nu\hat\nu F}\rangle - \sum_{\ell=1}^{2}\langle L^{(\ell)}_{\dot g}\mathrm{I\!I}^{\hat\nu}, A_{F\hat\nu\hat\nu F}\rangle, \tag{3.33}$$

we proceed to compute and simplify the first two terms on the right. Let $X, Y \in \mathfrak{X}(F)$ be time-independent vector fields. To compute the time derivative of the second fundamental form, we start by differentiating (2.7)

$$\begin{aligned}\frac{d}{dt}\mathrm{I\!I}^{\hat\nu}(X,Y) &= \frac{d}{dt}g(\hat\nu,\nabla_X Y) = \frac{1}{2}\dot g_{\hat\nu\hat\nu}g(\hat\nu,\nabla_X Y) + g\left(\hat\nu,\frac{d}{dt}\nabla_X Y\right)\\ &= \frac{1}{2}\dot g_{\hat\nu\hat\nu}\mathrm{I\!I}^{\hat\nu}(X,Y) + \frac{1}{2}\Big[(\nabla_X\dot g)(\hat\nu,Y) + (\nabla_Y\dot g)(\hat\nu,X) - (\nabla_{\hat\nu}\dot g)(X,Y)\Big],\end{aligned}$$

where we used (3.29) of Lemma 3.16 followed by (3.27). Noting that $A_{F\hat\nu\hat\nu F}(X,Y)$ and $(\nabla_X\dot g)(\hat\nu,Y) + (\nabla_Y\dot g)(\hat\nu,X) = (\nabla\dot g)_{F\hat\nu F}(X,Y) + (\nabla\dot g)_{F\hat\nu F}(Y,X)$ are both symmetric in X, Y, we obtain

$$\left\langle \dot{\mathrm{I\!I}}^{\hat\nu}, A_{F\hat\nu\hat\nu F}\right\rangle = \frac{1}{2}\dot g_{\hat\nu\hat\nu}\langle \mathrm{I\!I}^{\hat\nu}, A_{F\hat\nu\hat\nu F}\rangle + \langle(\nabla\dot g)_{F\hat\nu F} - \frac{1}{2}(\nabla\dot g)_{\hat\nu FF}, A_{F\hat\nu\hat\nu F}\rangle. \tag{3.34}$$

Next, note that for any time-independent $X, Y \in \mathfrak{X}(F)$,

$$\begin{aligned}\frac{d}{dt}A_{F\hat\nu\hat\nu F}(X,Y) &= \dot A_{F\hat\nu\hat\nu F}(X,Y) + A(X,\dot{\hat\nu},\hat\nu,Y) + A(X,\hat\nu,\dot{\hat\nu},Y)\\ &= \dot A_{F\hat\nu\hat\nu F}(X,Y) + \dot g_{\hat\nu\hat\nu}A_{F\hat\nu\hat\nu F}(X,Y) - A(X,L_{\dot g}\hat\nu,\hat\nu,Y) - A(X,\hat\nu,L_{\dot g}\hat\nu,Y)\\ &= (\dot g_{\hat\nu\hat\nu} - \operatorname{tr}\dot g)\,A_{F\hat\nu\hat\nu F}(X,Y) + A(L_{\dot g}X,\hat\nu,\hat\nu,Y) + A(X,\hat\nu,\hat\nu,L_{\dot g}Y),\end{aligned}$$

where the second and third equalities followed from Lemmas 3.16 and 3.13, respectively. Hence,

$$\left\langle \mathrm{I\!I}^{\hat\nu}, \frac{d}{dt}A_{F\hat\nu\hat\nu F}\right\rangle = \langle \mathrm{I\!I}^{\hat\nu}, (\dot g_{\hat\nu\hat\nu} - \operatorname{tr}\dot g)\,A_{F\hat\nu\hat\nu F} + 2L^{(1)}_{\dot g}A_{F\hat\nu\hat\nu F}\rangle,$$

using the symmetry of $\mathrm{I\!I}^{\hat\nu}$ and (3.17).

Finally, we return to (3.33) and use the above identity, together with (3.34) and (3.16), to obtain

$$\begin{aligned}\frac{d}{dt}\langle \mathrm{I\!I}^{\hat\nu}, A_{F\hat\nu\hat\nu F}\rangle = &\frac{1}{2}\dot g_{\hat\nu\hat\nu}\langle \mathrm{I\!I}^{\hat\nu}, A_{F\hat\nu\hat\nu F}\rangle + \langle(\nabla\dot g)_{F\hat\nu F} - \frac{1}{2}(\nabla\dot g)_{\hat\nu FF}, A_{F\hat\nu\hat\nu F}\rangle\\ &+ \langle \mathrm{I\!I}^{\hat\nu}, (\dot g_{\hat\nu\hat\nu} - \operatorname{tr}\dot g)\,A_{F\hat\nu\hat\nu F}\rangle.\end{aligned}$$

Since $\dot g_{\hat\nu\hat\nu} - \operatorname{tr}\dot g = -\operatorname{tr}(\dot g_F)$, the lemma now follows from the facet version of (3.14), namely $\dot\omega_F = \frac{1}{2}\operatorname{tr}(\dot g_F)\,\omega_F$. □

3.5. **Evolution of angle deficit.** Finally, we consider the time evolution of the last term (of codimension 2) of (2.15). Given $E \in \mathring{\mathscr{E}}$, let $\mathscr{F}_E = \{F \in \mathscr{F} : E \in \triangle_{-1}F\}$ denote the collection of facets sharing E. Each $F \in \mathscr{F}_E$ is shared by two mesh elements $T_\pm \in \mathscr{T}$. Recall that the previously defined g-conormal $\hat\mu_E^F$ and g-normals $\hat\nu_F^{T_\pm}$ (see Subsections 2.3–2.4 and Figure 1), are such that $\hat\nu_F^{T_\pm}$ points inward from F with respect to $T_\pm$ and $\hat\mu_E^F$ points outward from E into F. Using them, and letting $\sigma_\pm = \sigma|_{T_\pm}$, we define, for any $\sigma \in \mathcal{T}^2_0(\mathscr{T})$, a jump-like function on E by

$$[\![\sigma_{\hat\mu\hat\nu}]\!]^E_F = \sigma_+(\hat\nu_F^{T_+},\hat\mu_E^F) + \sigma_-(\hat\nu_F^{T_-},\hat\mu_E^F). \tag{3.35}$$

Also let σ_E denote σ restricted to act on tangential vector fields of E, i.e., for all $X, Y \in \mathfrak{X}(E)$, $\sigma_E(X,Y) = \sigma(X,Y)$.

Lemma 3.19. *There holds for each $E \in \mathring{\mathscr{E}}$ and $A \in \mathring{\mathbb{A}}$*

$$\frac{d}{dt}\left(\Theta_E A_{\hat\mu\hat\nu\hat\nu\hat\mu}\,\omega_E\right) = -\frac{1}{2}\left(\sum_{F\in\mathscr{F}_E}[\![\dot g_{\hat\nu\hat\mu}]\!]^E_F + \operatorname{tr}(\dot g_E)\Theta_E\right)A_{\hat\mu\hat\nu\hat\nu\hat\mu}\,\omega_E.$$

Proof. Denote by $\{\hat\tau_j\}_{j=1}^{N-2}$ the g-orthonormal frame of $\mathfrak{X}(E)$. As in the proof of Lemma 3.16, the time derivatives of $\hat\tau_j$ and $\hat\mu$ lie in their respective tangent spaces, i.e., $\dot{\hat\tau}_i \in \mathfrak{X}(E)$, $\dot{\hat\mu} \in \mathfrak{X}(F)$, and the identities

$$\begin{aligned} 0 &= \frac{d}{dt} g(\hat\mu, \hat\tau_i) = \dot g(\hat\mu, \hat\tau_i) + g(\dot{\hat\mu}, \hat\tau_i), \\ 0 &= \frac{d}{dt} g(\hat\mu, \hat\mu) = \dot g(\hat\mu, \hat\mu) + 2g(\dot{\hat\mu}, \hat\mu), \\ 0 &= \frac{d}{dt} g(\hat\mu, \hat\nu) = \dot g(\hat\mu, \hat\nu) + g(\hat\mu, \dot{\hat\nu}), \end{aligned}$$

imply that

$$\dot{\hat\mu} = \frac{1}{2}\dot g(\hat\mu, \hat\mu)\hat\mu - L_{\dot g}\hat\mu.$$

Hence, by Lemma 3.13, Lemma 3.16, and the symmetries (2.12) of A,

$$\begin{aligned} \frac{d}{dt} A_{\hat\mu\hat\nu\hat\nu\hat\mu} &= \dot A_{\hat\mu\hat\nu\hat\nu\hat\mu} + 2A(\dot{\hat\mu}, \hat\nu, \hat\nu, \hat\mu) + 2A(\hat\mu, \dot{\hat\nu}, \hat\nu, \hat\mu) \\ &= -\operatorname{tr}(\dot g) A_{\hat\mu\hat\nu\hat\nu\hat\mu} + 2A(L_{\dot g}\hat\mu, \hat\nu, \hat\nu, \hat\mu) + 2A(\hat\mu, L_{\dot g}\hat\nu, \hat\nu, \hat\mu) \\ &\quad + A(\dot g(\hat\mu, \hat\mu)\hat\mu - 2L_{\dot g}\hat\mu, \hat\nu, \hat\nu, \hat\mu) + A(\hat\mu, \dot g(\hat\nu, \hat\nu)\hat\nu - 2L_{\dot g}\hat\nu, \hat\nu, \hat\mu) \\ &= \big(\dot g(\hat\nu, \hat\nu) + \dot g(\hat\mu, \hat\mu) - \operatorname{tr}(\dot g)\big) A_{\hat\mu\hat\nu\hat\nu\hat\mu} = -\operatorname{tr}(\dot g_E) A_{\hat\mu\hat\nu\hat\nu\hat\mu}. \end{aligned}$$

Next recall that [21, Lemma 3.2] for the variation of the angle deficit (noting that in [21] the normal points outward leading to a different sign) gives

$$\frac{d}{dt}\Theta_E = -\frac{1}{2} \sum_{F \in \mathscr{F}_E} [\![\dot g_{\hat\nu\hat\mu}]\!]_F^E.$$

We also have the analogue of (3.14) for E, namely $\dot\omega_E = \frac{1}{2}\operatorname{tr}(\dot g_E)\,\omega_E$. Using each of these identities, the lemma follows after applying the Leibniz rule to differentiate $\Theta_E A_{\hat\mu\hat\nu\hat\nu\hat\mu}\omega_E$. □

Putting the lemmas together, we obtain the main result of this section, Theorem 3.20, below. There, the time derivative of the previously defined generalized densitized Riemann curvature is expressed in terms of two forms $a(g; \sigma, U)$ and $b(g; \sigma, U)$, both linear in $\sigma = \dot g$ and the metric-independent test function U, but nonlinear in the metric g. Note that $a(g; \sigma, U)$ does not contain any spatial derivatives nor jumps of σ, but $b(g; \sigma, U)$ does.

Theorem 3.20. *Let $\sigma = \dot g(t)$, $U \in \mathring{\mathfrak{U}}$ and $A = \mathbb{A}U \in \mathring{\mathcal{A}}$. Then*

$$\frac{d}{dt}\widetilde{\mathfrak{Q}\omega}(U) = a(g; \sigma, U) + b(g; \sigma, U), \tag{3.36}$$

where with $\mathbb{S}_F(\rho) = \rho_F - \operatorname{tr}(\rho_F) g_F$ *for all* $\rho \in \mathcal{S}(\mathscr{T})$

$$a(g;\sigma,U) := \sum_{T\in\mathscr{T}} \int_T \Big(\langle L_\sigma^{(1)} \mathcal{R},\ A\rangle - \frac{1}{2}\operatorname{tr}(\sigma)\,\langle \mathcal{R},\ A\rangle \Big)\omega_T$$

$$+ 2\sum_{F\in\mathring{\mathscr{F}}} \int_F [\![\mathrm{I\!I}]\!] : \mathbb{S}_F(\sigma) : A_{F\hat\nu\hat\nu F}\,\omega_F$$

$$- 2\sum_{E\in\mathring{\mathscr{E}}} \int_E \operatorname{tr}(\sigma_E)\,\Theta_E A_{\hat\mu\hat\nu\hat\nu\hat\mu}\,\omega_E, \tag{3.37}$$

$$b(g;\sigma,U) := 2\sum_{T\in\mathscr{T}} \int_T \langle \nabla^2\sigma,\ SA\rangle\,\omega_T$$

$$+ 2\sum_{F\in\mathring{\mathscr{F}}} \int_F \langle [\![\sigma_{\hat\nu\hat\nu}\mathrm{I\!I} + (\nabla\sigma)_{F\hat\nu F} + \nabla(\hat\nu \lrcorner \sigma) - (\nabla\sigma)_{\hat\nu FF}]\!],\ A_{F\hat\nu\hat\nu F}\rangle\omega_F$$

$$- 2\sum_{E\in\mathring{\mathscr{E}}} \int_E \sum_{F\in\mathscr{F}_E} [\![\sigma_{\hat\mu\hat\nu}]\!]_F^E\, A_{\hat\mu\hat\nu\hat\nu\hat\mu}\,\omega_E. \tag{3.38}$$

Proof. We differentiate each term of the identity of Theorem 3.4 and apply Lemmas 3.15, 3.18, and 3.19 for the terms on the right of codimension 0, 1, and 2, respectively. Then we apply Lemma 3.17 to rewrite the codimension 1 terms. Next, we set $\sigma = \dot g(t)$. Collecting all terms with spatial derivatives of σ and jumps of σ, we see that their sum equals b, while the remainder equals a. □

At this point, the grouping of terms into $a(\dots)$ and $b(\dots)$ may appear ad hoc. But, as we will show in the next section, the form $b(\dots)$ in (3.38) contains (up to a factor -2) a distributional version of a generalized covariant incompatibility operator of σ (see Theorem 4.1). In our numerical analysis of §5 we will estimate $a(g;\cdot,\cdot)$ and $b(g;\cdot,\cdot)$ independently. In two dimensions, $a(\dots)$ vanishes, as is shown in §6.1.

4. Distributional incompatibility operator and its adjoint

The incompatibility operator acting on a 2-tensor σ is well studied in two and three dimensional Euclidean domains. Motivated by the fact that in two dimensions it arises from the linearization of Gauss curvature, we propose a generalization of the incompatibility operator (in (4.4) below) to higher dimensions by examining the linearization of the Riemann curvature tensor. Taking this further (in Theorem 4.1 and Definition 4.5) we define a distributional generalization of incompatibility when σ is not smooth and belongs only to $\operatorname{Reg}(\mathscr{T})$. Finally, defining an adjoint of this generalized incompatibility operator (in Definition 4.6), the key result (Theorem 4.7) of this section is presented.

To define the generalized incompatibility of a smooth σ, first recall that the result of applying the projector P in (3.23) on any covariant 4-tensor A is

$$(PA)(X,Y,Z,W) = \frac{1}{4}\Big[A(X,Y,Z,W) - A(Z,Y,X,W) + A(Z,W,X,Y) - A(X,W,Z,Y)\Big]. \tag{4.1}$$

Since $P = P_{13}\circ P_{24} = P_{24}\circ P_{13}$, the result $B = PA$ after an application of P is skew symmetric in its first and third arguments as well as its second and fourth arguments. It is also immediate from (4.1) that $B(X,Y,Z,W) = B(Y,X,W,Z)$, i.e., $B = PA$ satisfies

$$B(X,Y,Z,W) = -B(Z,Y,X,W) = -B(X,W,Z,Y) = B(Y,X,W,Z). \tag{4.2}$$

The same symmetries can also be obtained from $\mathcal{A}$ using the operator S in (3.21). Namely, letting

$$\mathcal{B} = \{SA : A \in \mathcal{A}\}, \qquad \mathring{\mathcal{B}} = \{SA : A \in \mathring{\mathcal{A}}\},$$

we see that any $B \in \mathcal{B}$ satisfies (4.2). Note that tensors in $\mathcal{B}$ also have the additional continuity condition (2.13), i.e.,

$$B_{F\hat{\nu}\hat{\nu}F} = B_{\cdot\hat{\nu}\hat{\nu}\cdot} \tag{4.3}$$

is single-valued on interior facets $F \in \mathring{\mathscr{F}}$. (Note that the equality (4.3) follows from the skew symmetry (4.2) of B.) Let us denote by $\mathcal{D}\mathcal{A}$ and $\mathcal{D}\mathcal{B}$ the subspaces of $\mathcal{A} \cap \mathcal{T}_0^4(\Omega)$ and $\mathcal{B} \cap \mathcal{T}_0^4(\Omega)$ consisting of (globally smooth) compactly supported tensor fields.

From Lemma 3.14, we know that the linearization of the Riemann curvature tensor has a scalar multiple of $SP\nabla^2\sigma$ with $\sigma = \dot{g}$ when g is smooth. We relate incompatibility to this linearization, motivated by the known relationship between incompatibility and curvature linearization in two dimensions [23]. Namely, we define the covariant incompatibility of a smooth symmetric $\sigma \in \mathcal{S}(\Omega)$, when g is a globally smooth metric on Ω, by

$$\operatorname{Inc}\sigma := -SP\nabla^2\sigma. \tag{4.4}$$

Then, for any $A \in \mathcal{A}$, letting $B = SA$, the 4-tensor $\operatorname{Inc}\sigma$ satisfies

$$\langle \operatorname{Inc}\sigma, A\rangle = -\langle SP\nabla^2\sigma, A\rangle = -\langle \nabla^2\sigma, SA\rangle = -\langle \nabla^2\sigma, B\rangle \tag{4.5}$$

since, at each point on the manifold, $B = SA$ is in the range of P, and S is selfadjoint. Integrating (4.5) using the volume form ω of the smooth metric, we find that

$$\int_\Omega \langle \operatorname{Inc}\sigma, \Phi\rangle\,\omega = -\int_\Omega \langle \sigma, \operatorname{divdiv} S\Phi\rangle\,\omega, \qquad \Phi \in \mathcal{D}\mathcal{A}. \tag{4.6}$$

Here we have used integration by parts (see (A.16)) twice, recalling that Φ has compact support.

Next, suppose $\sigma \in \operatorname{Reg}(\mathscr{T})$. Then $\nabla^2\sigma$ is, in general, not definable as a classical derivative, and definition (4.4) needs extension. For this, we first define a linear functional $\widetilde{\nabla^2\sigma}$ on $\mathcal{D}\mathcal{B}$ by

$$\widetilde{\nabla^2\sigma}(\Psi) := \int_\Omega \langle \sigma, \operatorname{divdiv}\Psi\rangle\,\omega, \qquad \Psi \in \mathcal{D}\mathcal{B},\ \sigma \in \operatorname{Reg}(\mathscr{T}). \tag{4.7}$$

This generalizes the second covariant derivative's action on smooth tensors possessing the symmetries of $\mathcal{B}$. Using it, we extend $\operatorname{Inc}\sigma$, taking motivation from (4.5), as a linear functional $\widetilde{\operatorname{Inc}\sigma}$ acting on $\mathcal{D}\mathcal{A}$, as follows:

$$\widetilde{\operatorname{Inc}\sigma}(\Phi) := -\widetilde{\nabla^2\sigma}(S\Phi) = \int_\Omega \langle \sigma, \operatorname{divdiv} S\Phi\rangle\,\omega, \qquad \Phi \in \mathcal{D}\mathcal{A},\ \sigma \in \operatorname{Reg}(\mathscr{T}). \tag{4.8}$$

By (4.6), we see that this is indeed an extension of the smooth case (4.5). The next result characterizes this generalized incompatibility in terms of integrals over mesh components, including the smooth incompatibility computed element by element.

Theorem 4.1. *Let $\sigma \in \operatorname{Reg}(\mathscr{T})$, $\Phi \in \mathcal{D}\mathcal{A}$, and let $g \in \mathcal{S}^+(\Omega)$ be a smooth metric tensor. Then,*

$$\begin{aligned} \widetilde{\operatorname{Inc}\sigma}(\Phi) = \sum_{T\in\mathscr{T}} \Bigg(\int_T \langle \operatorname{Inc}\sigma, \Phi\rangle\, \omega_T \\ - \int_{\partial T} \langle \sigma_{\hat\nu\hat\nu} \mathrm{I\!I}^{\hat\nu} + (\nabla\sigma)_{F\hat\nu F} + \nabla(\hat\nu \lrcorner \sigma) - (\nabla\sigma)_{\hat\nu FF}, \Phi_{F\hat\nu\hat\nu F}\rangle\, \omega_{\partial T} \Bigg) \\ - \sum_{E\in\mathring{\mathscr{E}}} \sum_{F\in\mathscr{F}_E} \int_E [\![\sigma_{\hat\mu\hat\nu}]\!]_F^E \Phi_{\hat\mu\hat\nu\hat\nu\hat\mu}\, \omega_E, \end{aligned} \tag{4.9}$$

where, in the integrals over ∂T, the g-normal $\hat\nu$ points into the element T, and we have used the notation in (3.30) *and* (3.35)*. Hence, for any smooth compactly supported U in $\mathring{\mathcal{U}}$, letting $\Phi = \mathbb{A}U \in \mathcal{D}\mathring{\mathcal{A}}$, the form $b(\dots)$ in* (3.38) *satisfies*

$$b(g;\sigma,U) = -2\, \widetilde{\operatorname{Inc}\sigma}(\Phi). \tag{4.10}$$

We proceed to prove (4.9) of Theorem 4.1 using the next two lemmas. Identity (4.10) immediately follows from (4.9), but is stated to connect to the developments in the previous section by showing the relationship between the $b(\dots)$ in Theorem 3.20 and the just-defined distributional incompatibility operator of (4.8).

Lemma 4.2. *In the setting of Theorem 4.1, letting $\Psi = S\Phi \in \mathcal{D}\mathcal{B}$,*

$$\begin{aligned} \widetilde{\nabla^2\sigma}(\Psi) = \sum_{T\in\mathscr{T}} \Bigg(-\int_T \langle \nabla\sigma, \operatorname{div}\Psi\rangle\, \omega_T \\ + \int_{\partial T} \Big(\langle \sigma_{\hat\nu\hat\nu} \mathrm{I\!I}^{\hat\nu} + \nabla(\hat\nu\lrcorner\sigma),\ \Psi_{F\hat\nu\hat\nu F}\rangle - \langle \mathrm{I\!I}^{\hat\nu} \otimes \hat\nu^\flat \otimes \sigma_{\hat\nu F}, \Psi\rangle \Big)\, \omega_{\partial T} \Bigg) \\ + \sum_{F\in\mathring{\mathscr{F}}} \int_{\partial F} \langle [\![\sigma_{\hat\nu F}]\!],\ \Psi_{\hat\mu\hat\nu\hat\nu F}\rangle\, \omega_{\partial F}, \end{aligned} \tag{4.11}$$

where, in the last term, the g-conormal vector $\hat\mu$ on ∂F points into the facet F.

Proof. Starting from (4.7) and integrating by parts elementwise using (A.16),

$$\begin{aligned} \widetilde{\nabla^2\sigma}(\Psi) &= \sum_{T\in\mathscr{T}} \int_T \langle \sigma, \operatorname{div}\operatorname{div}\Psi\rangle\, \omega_T \\ &= \sum_{T\in\mathscr{T}} \Bigg(-\int_T \langle\nabla\sigma, \operatorname{div}\Psi\rangle\, \omega_T - \int_{\partial T} \langle \hat\nu^\flat \otimes \sigma, \operatorname{div}\Psi\rangle\, \omega_{\partial T} \Bigg). \end{aligned} \tag{4.12}$$

Then using the notation in (2.3) to split the last integrand using normal and tangential components,

$$\begin{aligned} \langle \hat\nu^\flat \otimes \sigma,\ \operatorname{div}\Psi\rangle = \langle \sigma_{\hat\nu\hat\nu}, (\operatorname{div}\Psi)_{\hat\nu\hat\nu\hat\nu}\rangle + \langle \sigma_{F\hat\nu}, (\operatorname{div}\Psi)_{\hat\nu F\hat\nu}\rangle \\ + \langle \sigma_{\hat\nu F}, (\operatorname{div}\Psi)_{\hat\nu\hat\nu F}\rangle + \langle \sigma_F, (\operatorname{div}\Psi)_{\hat\nu FF}\rangle. \end{aligned} \tag{4.13}$$

Since Ψ is skew symmetric in its second and fourth arguments—see (4.2)—we know that $\operatorname{div}\Psi$ is skew symmetric in its first and third arguments (as can be seen from (A.13)), so $(\operatorname{div}\Psi)_{\hat\nu\hat\nu\hat\nu}$ and $(\operatorname{div}\Psi)_{\hat\nu F\hat\nu}$ vanish. The tt-continuity of σ implies that the σ_F term in (4.13) is exactly the same from the adjacent element, except for a sign. Using all these observations (and the jump notation in (2.9)), we find that (4.13) implies that at any point of an interior facet,

$$\langle [\![\hat\nu^\flat \otimes \sigma]\!], \operatorname{div}\Psi\rangle = \langle [\![\sigma_{\hat\nu F}]\!], (\operatorname{div}\Psi)_{\hat\nu\hat\nu F}\rangle = \langle [\![\sigma_{\hat\nu F}]\!], (\operatorname{div}^F\Psi)_{\hat\nu\hat\nu F}\rangle.$$

Here we have used that, by (A.17), the surface divergence satisfies $(\mathrm{div}^F\Psi)_{\hat\nu\hat\nu F} = (\mathrm{div}\Psi)_{\hat\nu\hat\nu F} - (\nabla\Psi)_{\hat\nu\hat\nu\hat\nu\hat\nu F}$ where the last term vanishes due to skew symmetries of Ψ. Moreover, since $(\mathrm{div}^F\Psi)_{\hat\nu\hat\nu\hat\nu}$ vanishes and since

$$\hat\nu\lrcorner\sigma = \sigma_{\hat\nu\hat\nu}\hat\nu^\flat + \sigma_{\hat\nu F}, \tag{4.14}$$

we may rewrite $\langle[\![\sigma_{\hat\nu F}]\!], (\mathrm{div}^F\Psi)_{\hat\nu\hat\nu F}\rangle = \langle\hat\nu^\flat\otimes\hat\nu^\flat\otimes[\![\hat\nu\lrcorner\sigma]\!], \mathrm{div}^F\Psi\rangle$. Accounting for all the above-mentioned cancellations, (4.12) becomes

$$\widetilde{\nabla^2\sigma}(\Psi) = \sum_{T\in\mathscr{T}}\left(-\int_T\langle\nabla\sigma, \mathrm{div}\Psi\rangle\omega_T - \int_{\partial T}\langle\hat\nu^\flat\otimes\hat\nu^\flat\otimes(\hat\nu\lrcorner\sigma),\, \mathrm{div}^F\Psi\rangle\omega_{\partial T}\right). \tag{4.15}$$

The last term is now in a form suitable for integration by parts on each facet F of ∂T using (A.22). Doing so, we get

$$\begin{aligned} -\int_F\langle\hat\nu^\flat\otimes\hat\nu^\flat\otimes(\hat\nu\lrcorner\sigma),\, \mathrm{div}^F\Psi\rangle\omega_F &= \int_F H^{\hat\nu}\,\langle\hat\nu^\flat\otimes\hat\nu^\flat\otimes\hat\nu^\flat\otimes(\hat\nu\lrcorner\sigma),\, \Psi\rangle\omega_F \\ &+ \int_F\langle\nabla^F(\hat\nu^\flat\otimes\hat\nu^\flat\otimes(\hat\nu\lrcorner\sigma)), \Psi\rangle\omega_F + \int_{\partial F}\langle\hat\mu^\flat\otimes\hat\nu^\flat\otimes\hat\nu^\flat\otimes(\hat\nu\lrcorner\sigma),\, \Psi\rangle\omega_{\partial F}, \end{aligned} \tag{4.16}$$

where $\hat\mu$ denotes the conormal vector on ∂F pointing into F, $H^{\hat\nu}$ denotes the mean curvature (see (A.19)), and ∇^F is as in (A.18). By the skew symmetry of Ψ in its first and third arguments, $\Psi(\hat\nu,\hat\nu,\hat\nu,X) = 0$ for all $X\in\mathfrak{X}(\mathscr{T})$, so the integral with $H^{\hat\nu}$ vanishes.

To simplify the next integral with ∇^F in (4.16), we begin by noting that (A.18) implies $\nabla^F A(\hat\nu,\dots) = 0$ for any tensor A. Hence the inner product in the integrand can be evaluated using a g-orthonormal basis $\{\hat\tau_i\}_{i=1}^{N-1}$ of the tangent space $\mathfrak{X}(F)$, i.e.,

$$\begin{aligned} \langle\nabla^F(\hat\nu^\flat\otimes\hat\nu^\flat\otimes(\hat\nu\lrcorner\sigma)), \Psi\rangle &= \sum_{i=1}^{N-1}\langle\nabla^F_{\hat\tau_i}(\hat\nu^\flat\otimes\hat\nu^\flat\otimes(\hat\nu\lrcorner\sigma)), \hat\tau_i\lrcorner\Psi\rangle \\ = \sum_{i=1}^{N-1}\langle(\nabla^F_{\hat\tau_i}\hat\nu^\flat)\otimes\hat\nu^\flat\otimes(\hat\nu\lrcorner\sigma) &+ \hat\nu^\flat\otimes(\nabla^F_{\hat\tau_i}\hat\nu^\flat)\otimes(\hat\nu\lrcorner\sigma) + \hat\nu^\flat\otimes\hat\nu^\flat\otimes\nabla^F_{\hat\tau_i}(\hat\nu\lrcorner\sigma), \hat\tau_i\lrcorner\Psi\rangle, \end{aligned} \tag{4.17}$$

where we have also used the Leibniz rule. Note that $\nabla^F\hat\nu^\flat = -\mathrm{I\!I}^{\hat\nu}$ can be expressed in terms of $\hat\tau_i^\flat\otimes\hat\tau_j^\flat$ for all i,j (without $\hat\nu$). Also using (4.14), the first term in (4.17) becomes

$$\begin{aligned} &\sum_{i=1}^{N-1}\langle(\nabla^F_{\hat\tau_i}\hat\nu^\flat)\otimes\hat\nu^\flat\otimes(\hat\nu\lrcorner\sigma), \hat\tau_i\lrcorner\Psi\rangle \\ &\quad= \langle\nabla^F\hat\nu^\flat\otimes\hat\nu^\flat\otimes\sigma_{\hat\nu F},\, \Psi\rangle + \sum_{i=1}^{N-1}\sigma_{\hat\nu\hat\nu}\langle(\nabla^F_{\hat\tau_i}\hat\nu^\flat)\otimes\hat\nu^\flat\otimes\hat\nu^\flat,\, \hat\tau_i\lrcorner\Psi\rangle \\ &\quad= -\langle\mathrm{I\!I}^{\hat\nu}\otimes\hat\nu\otimes\sigma_{\hat\nu F}, \Psi\rangle - \sum_{j=1}^{N-1}\sum_{i=1}^{N-1}\sigma_{\hat\nu\hat\nu}\,\mathrm{I\!I}^{\hat\nu}(\hat\tau_i,\hat\tau_j)\,\Psi(\hat\tau_i,\hat\tau_j,\hat\nu,\hat\nu) \\ &\quad= -\langle\mathrm{I\!I}^{\hat\nu}\otimes\hat\nu\otimes\sigma_{\hat\nu F}, \Psi\rangle + \langle\sigma_{\hat\nu\hat\nu}\,\mathrm{I\!I}^{\hat\nu}, \Psi_{F\hat\nu\hat\nu F}\rangle, \end{aligned}$$

since $\Psi(\hat\tau_i,\hat\tau_j,\hat\nu,\hat\nu) = -\Psi(\hat\tau_i,\hat\nu,\hat\nu,\hat\tau_j)$. Using the skew symmetry of $\psi_{ij}(X) = \Psi(\hat\tau_i,\hat\nu,\hat\tau_j,X)$ for all $X\in\mathfrak{X}(\mathscr{T})$ with respect to i,j and the symmetry of $\mathrm{I\!I}^{\hat\nu}(\hat\tau_i,\hat\tau_j)$

with respect to i, j, we find that the second term in (4.17) must vanish

$$-\sum_{i=1}^{N-1}\langle\hat{\nu}^\flat\otimes(\nabla^F_{\hat{\tau}_i}\hat{\nu}^\flat)\otimes(\hat{\nu}\lrcorner\sigma),\hat{\tau}_i\lrcorner\Psi\rangle=\sum_{j=1}^{N-1}\sum_{i=1}^{N-1}\langle\mathbb{I}^{\hat{\nu}}(\hat{\tau}_i,\hat{\tau}_j)\,(\hat{\nu}\lrcorner\sigma),\,\psi_{ij}\rangle=0.$$

Thus (4.17) becomes

$$\langle\nabla^F(\hat{\nu}^\flat\otimes\hat{\nu}^\flat\otimes(\hat{\nu}\lrcorner\sigma)),\Psi\rangle=\langle\sigma_{\hat{\nu}\hat{\nu}}\mathbb{I}^{\hat{\nu}}+\nabla^F(\hat{\nu}\lrcorner\sigma),\Psi_{F\hat{\nu}\hat{\nu}F}\rangle-\langle\mathbb{I}^{\hat{\nu}}\otimes\hat{\nu}^\flat\otimes\sigma_{\hat{\nu}F},\Psi\rangle. \tag{4.18}$$

Finally, using (4.18) in (4.16) and substituting the result into (4.15),

$$\widetilde{\nabla^2\sigma}(\Psi)=\sum_{T\in\mathscr{T}}\Big[-\int_T\langle\nabla\sigma,\operatorname{div}\Psi\rangle\omega_T+\int_{\partial T}\Big(\langle\sigma_{\hat{\nu}\hat{\nu}}\mathbb{I}^{\hat{\nu}}+\nabla^F(\hat{\nu}\lrcorner\sigma),\Psi_{F\hat{\nu}\hat{\nu}F}\rangle$$
$$-\langle\mathbb{I}^{\hat{\nu}}\otimes\hat{\nu}^\flat\otimes\sigma_{\hat{\nu}F},\Psi\rangle\Big)\omega_{\partial T}+\sum_{\{F\in\mathscr{F}:F\subset\partial T\}}\int_{\partial F}\langle\hat{\mu}^\flat\otimes\hat{\nu}^\flat\otimes\hat{\nu}^\flat\otimes(\hat{\nu}\lrcorner\sigma),\,\Psi\rangle\omega_{\partial F}\Big].$$

The last integrand, upon use of the decomposition (4.14), becomes a sum of two terms, $\langle\hat{\mu}^\flat\otimes\hat{\nu}^\flat\otimes\hat{\nu}^\flat\otimes(\hat{\nu}\lrcorner\sigma),\,\Psi\rangle=\langle\sigma_{\hat{\nu}F},\,\Psi_{\hat{\mu}\hat{\nu}\hat{\nu}F}\rangle+\langle\sigma_{\hat{\nu}\hat{\nu}},\,\Psi_{\hat{\mu}\hat{\nu}\hat{\nu}\hat{\nu}}\rangle$, of which the latter vanishes due to the skew symmetric properties of Ψ. This gives the stated result. ☐

Lemma 4.3. *In the setting of Theorem 4.1, letting* $\Psi=S\Phi$,

$$\sum_{F\in\mathring{\mathscr{F}}}\int_{\partial F}\langle[\![\sigma_{\hat{\nu}F}]\!],\Psi_{\hat{\mu}\hat{\nu}\hat{\nu}F}\rangle\omega_{\partial F}=\sum_{E\in\mathring{\mathscr{E}}}\sum_{F\in\mathscr{F}_E}\int_E[\![\sigma_{\hat{\nu}\hat{\mu}}]\!]^E_F\Psi_{\hat{\mu}\hat{\nu}\hat{\nu}\hat{\mu}}\,\omega_E.$$

Proof. This proof extends an idea of [13] from 3D to arbitrary dimension. Due to the tt-continuity of σ and the skew-symmetry of Ψ there holds with the projection P (3.23)

$$\langle[\![\hat{\nu}\lrcorner\sigma]\!],\Psi_{\hat{\mu}\hat{\nu}\hat{\nu}}\rangle=\langle[\![\hat{\nu}\lrcorner\sigma]\!],\Psi_{\hat{\mu}\hat{\nu}\hat{\nu}}\rangle+\langle[\![\sigma_F]\!],\Psi_{\hat{\mu}\hat{\nu}FF}\rangle+\langle[\![\sigma_{F\hat{\nu}}]\!],\Psi_{\hat{\mu}\hat{\nu}F\hat{\nu}}\rangle$$
$$=\langle[\![\sigma]\!],\Psi_{\hat{\mu}\hat{\nu}}\rangle=\langle P(\hat{\mu}^\flat\otimes\hat{\nu}^\flat\otimes[\![\sigma]\!]),\Psi\rangle$$

and thus,

$$\sum_{F\in\mathring{\mathscr{F}}}\int_{\partial F}\langle[\![\hat{\nu}\lrcorner\sigma]\!],\Psi_{\hat{\mu}\hat{\nu}\hat{\nu}}\rangle\omega_{\partial F}=\sum_{F\in\mathring{\mathscr{F}}}\int_{\partial F}\langle P(\hat{\mu}^\flat\otimes\hat{\nu}^\flat\otimes[\![\sigma]\!]),\Psi\rangle\omega_{\partial F}. \tag{4.19}$$

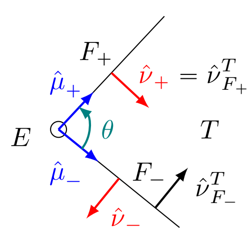

Figure 2. Illustration of g-normal and g-conormal vectors in the proof of Lemma 4.3.

First, we focus on $\hat{\mu}^\flat\otimes\hat{\nu}^\flat\otimes[\![\sigma]\!]$ by reordering the sum over codimension 2 boundaries. Let $\{\hat{\tau}_i\}_{i=1}^{N-2}$ be a g-orthonormal basis of $\mathfrak{X}(E)$. For each element T containing E as the intersection of the two facets F_+, $F_-\in\triangle_{-1}T$ we choose the orientation $(\hat{\nu}_+, \hat{\mu}_+)$ and $(\hat{\nu}_-, \hat{\mu}_-)$ such that both build a right-handed orthonormal basis. Without loss of generality we assume that $\hat{\nu}_+ = \hat{\nu}^T_{F_+}$ and $\hat{\nu}_- = -\hat{\nu}^T_{F_-}$, cf. Figure 2 and [13]. Then,

$$\sum_{F\in\mathring{\mathscr{F}}} \hat{\mu}^\flat \otimes \hat{\nu}^\flat \otimes [\![\sigma]\!] = \sum_{E\in\mathring{\mathscr{E}}} \sum_{T\supset E} \hat{\mu}^\flat_+ \otimes \hat{\nu}^\flat_+ \otimes \sigma|_T - \hat{\mu}^\flat_- \otimes \hat{\nu}^\flat_- \otimes \sigma|_T.$$

Let θ denote the angle to transform $(\hat{\nu}_-, \hat{\mu}_-)$ into $(\hat{\nu}_+, \hat{\mu}_+)$ and $R(\rho)$ the rotation matrix such that $R(0)\hat{\nu}_- = \hat{\nu}_-$, $R(0)\hat{\mu}_- = \hat{\mu}_-$ and $R(\theta)\hat{\nu}_- = \hat{\nu}_+$, $R(\theta)\hat{\mu}_- = \hat{\mu}_+$. Then

$$\hat{\mu}^\flat_+ \otimes \hat{\nu}^\flat_+ \otimes \sigma|_T - \hat{\mu}^\flat_- \otimes \hat{\nu}^\flat_- \otimes \sigma|_T = \int_0^\theta \frac{d}{d\rho}\big(R(\rho)\hat{\mu}^\flat_- \otimes R(\rho)\hat{\nu}^\flat_- \otimes \sigma|_T\big)\, d\rho.$$

Computing

$$\frac{d}{d\rho}\big(R(\rho)\hat{\mu}^\flat_- \otimes R(\rho)\hat{\nu}^\flat_- \otimes \sigma|_T\big) = -R(\rho)\hat{\nu}^\flat_- \otimes R(\rho)\hat{\nu}^\flat_- \otimes \sigma|_T + R(\rho)\hat{\mu}^\flat_- \otimes R(\rho)\hat{\mu}^\flat_- \otimes \sigma|_T$$

reveals that the integrand, and thus also the integral, is symmetric in its first two components. Therefore, we obtain the symmetry

$$\sum_{F\in\mathring{\mathscr{F}}} P(\hat{\mu}^\flat \otimes \hat{\nu}^\flat \otimes [\![\sigma]\!]) = \sum_{F\in\mathring{\mathscr{F}}} P(\hat{\nu}^\flat \otimes \hat{\mu}^\flat \otimes [\![\sigma]\!]). \tag{4.20}$$

Next, we expand $[\![\sigma]\!]$ into (co)normal and tangential components and use the skew-symmetry of P and tt-continuity of σ (only here, we implicitly sum over i, j if applicable)

$$\begin{aligned}
P(\hat{\mu}^\flat \otimes \hat{\nu}^\flat \otimes [\![\sigma]\!]) &= [\![\sigma(\hat{\tau}_i, \hat{\tau}_j)]\!] P(\hat{\mu}^\flat \otimes \hat{\nu}^\flat \otimes \hat{\tau}^\flat_i \otimes \hat{\tau}^\flat_j) + [\![\sigma(\hat{\nu}, \hat{\nu})]\!] P(\hat{\mu}^\flat \otimes \hat{\nu}^\flat \otimes \hat{\nu}^\flat \otimes \hat{\nu}^\flat)\\
&\quad + [\![\sigma(\hat{\mu}, \hat{\mu})]\!] P(\hat{\mu}^\flat \otimes \hat{\nu}^\flat \otimes \hat{\mu}^\flat \otimes \hat{\mu}^\flat) + [\![\sigma(\hat{\mu}, \hat{\nu})]\!] P(\hat{\mu}^\flat \otimes \hat{\nu}^\flat \otimes \hat{\mu}^\flat \otimes \hat{\nu}^\flat)\\
&\quad + [\![\sigma(\hat{\nu}, \hat{\mu})]\!] P(\hat{\mu}^\flat \otimes \hat{\nu}^\flat \otimes \hat{\nu}^\flat \otimes \hat{\mu}^\flat) + [\![\sigma(\hat{\mu}, \hat{\tau}_i)]\!] P(\hat{\mu}^\flat \otimes \hat{\nu}^\flat \otimes \hat{\mu}^\flat \otimes \hat{\tau}^\flat_i)\\
&\quad + [\![\sigma(\hat{\tau}_i, \hat{\mu})]\!] P(\hat{\mu}^\flat \otimes \hat{\nu}^\flat \otimes \hat{\tau}^\flat_i \otimes \hat{\mu}^\flat) + [\![\sigma(\hat{\nu}, \hat{\tau}_i)]\!] P(\hat{\mu}^\flat \otimes \hat{\nu}^\flat \otimes \hat{\nu}^\flat \otimes \hat{\tau}^\flat_i)\\
&\quad + [\![\sigma(\hat{\tau}_i, \hat{\nu})]\!] P(\hat{\mu}^\flat \otimes \hat{\nu}^\flat \otimes \hat{\tau}^\flat_i \otimes \hat{\nu}^\flat)\\
&= [\![\sigma(\hat{\nu}, \hat{\mu})]\!] P(\hat{\mu}^\flat \otimes \hat{\nu}^\flat \otimes \hat{\nu}^\flat \otimes \hat{\mu}^\flat) + [\![\sigma(\hat{\nu}, \hat{\tau}_i)]\!] P(\hat{\mu}^\flat \otimes \hat{\nu}^\flat \otimes \hat{\nu}^\flat \otimes \hat{\tau}^\flat_i),
\end{aligned}$$

and analogously

$$P(\hat{\nu}^\flat \otimes \hat{\mu}^\flat \otimes [\![\sigma]\!]) = [\![\sigma(\hat{\mu}, \hat{\nu})]\!] P(\hat{\nu}^\flat \otimes \hat{\mu}^\flat \otimes \hat{\mu}^\flat \otimes \hat{\nu}^\flat) + [\![\sigma(\hat{\tau}_i, \hat{\nu})]\!] P(\hat{\nu}^\flat \otimes \hat{\mu}^\flat \otimes \hat{\tau}^\flat_i \otimes \hat{\nu}^\flat).$$

Due to the proven symmetry (4.20) we have with the symmetries of σ and P

$$\begin{aligned}
0 &= \sum_{F\in\mathring{\mathscr{F}}} \Big(P(\hat{\mu}^\flat \otimes \hat{\nu}^\flat \otimes [\![\sigma]\!]) - P(\hat{\nu}^\flat \otimes \hat{\mu}^\flat \otimes [\![\sigma]\!])\Big)\\
&= \sum_{F\in\mathring{\mathscr{F}}} \sum_{i=1}^{N-2} [\![\sigma(\hat{\nu}, \hat{\tau}_i)]\!] \Big(P(\hat{\mu}^\flat \otimes \hat{\nu}^\flat \otimes \hat{\nu}^\flat \otimes \hat{\tau}^\flat_i) - P(\hat{\nu}^\flat \otimes \hat{\nu}^\flat \otimes \hat{\tau}^\flat_i \otimes \hat{\mu}^\flat)\Big)\\
&= \sum_{F\in\mathring{\mathscr{F}}} \sum_{i=1}^{N-2} [\![\sigma(\hat{\nu}, \hat{\tau}_i)]\!] P\Big(\hat{\mu}^\flat \otimes \hat{\nu}^\flat \otimes \hat{\nu}^\flat \otimes \hat{\tau}^\flat_i - \hat{\tau}^\flat_i \otimes \hat{\nu}^\flat \otimes \hat{\nu}^\flat \otimes \hat{\mu}^\flat\Big),
\end{aligned}$$

and thus there must holds

$$\sum_{F\in\mathring{\mathscr{F}}} \sum_{i=1}^{N-2} [\![\sigma(\hat{\nu}, \hat{\tau}_i)]\!] P(\hat{\mu}^\flat \otimes \hat{\nu}^\flat \otimes \hat{\nu}^\flat \otimes \hat{\tau}^\flat_i) = 0.$$

Using this identity in (4.19) gives the desired result

$$\begin{aligned}\sum_{F\in\mathring{\mathscr{F}}}\int_{\partial F}\langle[\![\hat\nu\lrcorner\sigma]\!],\Psi_{\hat\mu\hat\nu\hat\nu}\rangle\omega_{\partial F} &= \sum_{F\in\mathring{\mathscr{F}}}\int_{\partial F}\Big(\langle[\![\sigma(\hat\nu,\hat\mu)]\!]\hat\mu^\flat\otimes\hat\nu^\flat\otimes\hat\nu^\flat\otimes\hat\mu^\flat \\ &\qquad+\sum_{i=1}^{N-2}[\![\sigma(\hat\nu,\hat\tau_i)]\!]\hat\mu^\flat\otimes\hat\nu^\flat\otimes\hat\nu^\flat\otimes\hat\tau_i^\flat,\Psi\rangle\Big)\,\omega_{\partial F}\\ &=\sum_{E\in\mathring{\mathscr{E}}}\sum_{F\supset E}\int_E[\![\sigma(\hat\nu,\hat\mu)]\!]_F^E\Psi_{\hat\mu\hat\nu\hat\nu\hat\mu}\,\omega_E,\end{aligned}$$

finishing the proof. □

Proof of Theorem 4.1. Let $\Psi = S\Phi \in \mathcal{DB}$. We start by integrating by parts the first term of (4.11) (in Lemma 4.2) using (A.16),

$$\int_T -\langle\nabla\sigma,\mathrm{div}\Psi\rangle\omega_T = \int_T\langle\nabla^2\sigma,\Psi\rangle\omega_T + \int_{\partial T}\langle\hat\nu^\flat\otimes\nabla\sigma,\Psi\rangle\omega_{\partial T}. \tag{4.21}$$

Splitting the boundary integrand into normal and tangential components and omitting terms that vanish by the skew symmetries (4.2) of Ψ,

(4.22)

$$\langle\hat\nu^\flat\otimes\nabla\sigma,\Psi\rangle = \langle(\nabla\sigma)_{\hat\nu FF},\Psi_{\hat\nu\hat\nu FF}\rangle + \langle(\nabla\sigma)_{FF\hat\nu},\Psi_{\hat\nu FF\hat\nu}\rangle + \langle(\nabla\sigma)_{FFF},\Psi_{\hat\nu FFF}\rangle.$$

Since the first two terms on the right are present in the identity of the theorem, we focus on the last term. It can be understood by splitting σ into normal and tangential components,

$$\sigma = \hat\nu^\flat\otimes\sigma_{\hat\nu F} + \sigma_{F\hat\nu}\otimes\hat\nu^\flat + \sigma_{\hat\nu\hat\nu}\hat\nu^\flat\otimes\hat\nu^\flat + \sigma_F. \tag{4.23}$$

To compute $(\nabla^F\sigma)_F = (\nabla\sigma)_{FFF}$ using this decomposition, we start by applying the Leibniz rule to the first term on the right-hand side. Then, using $\mathrm{I\!I}^{\hat\nu} = -\nabla^F\hat\nu^\flat$ and (4.14), $\big(\nabla^F(\hat\nu^\flat\otimes\sigma_{\hat\nu F})\big)_F = \big(\hat\nu^\flat\otimes\nabla^F\sigma_{\hat\nu F}\big)_F - \mathrm{I\!I}^{\hat\nu}\otimes\sigma_{\hat\nu F} = -\mathrm{I\!I}^{\hat\nu}\otimes\sigma_{\hat\nu F}$, so

$$\langle\big(\nabla^F(\hat\nu^\flat\otimes\sigma_{\hat\nu F})\big)_F,\Psi_{\hat\nu FFF}\rangle = \langle-\mathrm{I\!I}^{\hat\nu}\otimes\sigma_{\hat\nu F},-\Psi_{FF\hat\nu F}\rangle = \langle\mathrm{I\!I}^{\hat\nu}\otimes\hat\nu^\flat\otimes\sigma_{\hat\nu F},\Psi\rangle.$$

The derivative of the second term on the right-hand side of (4.23) is computed similarly, but carefully accounting for the order of arguments, namely for $X_i \in \mathfrak{X}(F)$,

$$\begin{aligned}\big(\nabla^F(\sigma_{F\hat\nu}\otimes\hat\nu^\flat)\big)_F(X_1,X_2,X_3) &= \big(\nabla^F_{X_1}\big(\sigma_{F\hat\nu}\otimes\hat\nu^\flat\big)\big)(X_2,X_3)\\ &= \big(\nabla^F_{X_1}\sigma_{F\hat\nu}\otimes\hat\nu^\flat + \sigma_{F\hat\nu}\otimes\nabla^F_{X_1}\hat\nu^\flat\big)(X_2,X_3)\\ &= -\sigma_{\hat\nu F}(X_2)\,\mathrm{I\!I}^{\hat\nu}(X_1,X_3),\end{aligned}$$

since $\hat\nu^\flat(X_3) = 0$. As a result, by the symmetry of $\mathrm{I\!I}^{\hat\nu}(X_1,X_3)$ and the skew symmetry of $\Psi(\hat\nu,X_1,X_2,X_3)$ in X_1 and X_3,

$$\langle\nabla^F\big(\sigma_{F\hat\nu}\otimes\hat\nu^\flat\big)_F,\Psi_{\hat\nu FFF}\rangle = 0.$$

The third term in (4.23) does not contribute since $(\nabla^F(\sigma_{\hat\nu\hat\nu}\hat\nu^\flat\otimes\hat\nu^\flat))_F = 0$.

Using these observations to simplify the last term of (4.22), we obtain

$$\sum_{T\in\mathscr{T}}\int_{\partial T}\langle\hat\nu^\flat\otimes\nabla\sigma,\Psi\rangle\omega_{\partial T}=\sum_{T\in\mathscr{T}}\int_{\partial T}\Big(\langle(\nabla\sigma)_{F\hat\nu F},\Psi_{\hat\nu FF\hat\nu}\rangle-\langle(\nabla\sigma)_{\hat\nu FF},\Psi_{\hat\nu FF\hat\nu}\rangle$$
$$+\langle \mathrm{I\!I}^{\hat\nu}\otimes\hat\nu^\flat\otimes\sigma_{\hat\nu F},\Psi\rangle\Big)\omega_{\partial T}+\sum_{F\in\mathring{\mathscr{F}}}\int_F\langle[\![\sigma_F]\!],\Psi_{\hat\nu FFF}\rangle\omega_F.$$

By the *tt*-continuity of σ, the last term vanishes. The penultimate term appears with the opposite sign in the expression for $\widetilde{\nabla^2\sigma}(\Psi)$ in Lemma 4.2. That expression, put together with (4.21) and the above, gives

$$\widetilde{\nabla^2\sigma}(\Psi)=\sum_{T\in\mathscr{T}}\Big[\int_T\langle\nabla^2\sigma,\Psi\rangle\omega_T$$
$$+\int_{\partial T}\Big(\langle(\nabla\sigma)_{F\hat\nu F}-(\nabla\sigma)_{\hat\nu FF},\Psi_{F\hat\nu\hat\nu F}\rangle+\langle \mathrm{I\!I}^{\hat\nu}\otimes\hat\nu^\flat\otimes\sigma_{\hat\nu F},\Psi\rangle\Big)\omega_{\partial T}$$
$$+\int_{\partial T}\Big(\langle\sigma_{\hat\nu\hat\nu}\mathrm{I\!I}^{\hat\nu}+\nabla(\hat\nu\lrcorner\sigma),\ \Psi_{F\hat\nu\hat\nu F}\rangle-\langle \mathrm{I\!I}^{\hat\nu}\otimes\hat\nu^\flat\otimes\sigma_{\hat\nu F},\Psi\rangle\Big)\omega_{\partial T}\Big]$$
$$+\sum_{F\in\mathring{\mathscr{F}}}\int_{\partial F}\langle[\![\sigma_{\hat\nu F}]\!],\ \Psi_{\hat\mu\hat\nu\hat\nu F}\rangle\omega_{\partial F}.$$

Using Lemma 4.3 for the integrands over codimension 2 boundary terms, recalling that $\widetilde{\operatorname{Inc}\sigma}(\Phi)=-\widetilde{\nabla^2\sigma}(\Psi)$ by (4.8), and noting that the definition of S implies that $\Phi_{F\hat\nu\hat\nu F}=\Psi_{F\hat\nu\hat\nu F}$ and $\Phi_{\hat\mu\hat\nu\hat\nu\hat\mu}=\Psi_{\hat\mu\hat\nu\hat\nu\hat\mu}$, we prove (4.9). The proof of (4.10) follows by comparing the terms in (4.9) and (3.38) and applying (4.5). □

Remark 4.4. In Theorem 4.1, assuming that the metric g is smooth, we obtained (4.10), namely $b(g;\sigma,U)=-2\,\widetilde{\operatorname{Inc}\sigma}(\Phi)$, for smooth U and $\Phi=\mathbb{A}U$. However, all terms in (3.38) defining $b(g;\sigma,U)$ make sense even when Φ is less smooth, as long as Φ has the continuity properties of (2.13). This motivates us to conjecture that Theorem 4.1 also holds for $B\in\mathring{\mathcal{B}}$. To prove this rigorously, it suffices to show that smooth functions are dense in $\mathcal{B}$, which is an interesting question on its own, but not needed for the current analysis of the correctness of our generalized Riemann curvature expression. Instead, all we need, for now, is the following definition, which just amounts to a renaming of $b(\dots)$.

Definition 4.5. For any $\sigma\in\operatorname{Reg}(\mathscr{T})$, define $\widetilde{\operatorname{Inc}\sigma}$ as a linear functional on $\mathring{\mathcal{A}}$ by

$$\widetilde{\operatorname{Inc}\sigma}(A)=\sum_{T\in\mathscr{T}}\Big(\int_T\langle\operatorname{Inc}\sigma,A\rangle\omega_T$$
$$-\int_{\partial T}\langle\sigma_{\hat\nu\hat\nu}\mathrm{I\!I}^{\hat\nu}+(\nabla\sigma)_{F\hat\nu F}+\nabla(\hat\nu\lrcorner\sigma)-(\nabla\sigma)_{\hat\nu FF},A_{F\hat\nu\hat\nu F}\rangle\omega_{\partial T}\Big)$$
$$-\sum_{E\in\mathring{\mathscr{E}}}\sum_{F\in\mathscr{F}_E}\int_E[\![\sigma_{\hat\mu\hat\nu}]\!]^E_F A_{\hat\mu\hat\nu\hat\nu\hat\mu}\,\omega_E,$$

where the jump in the last term is as defined in (3.35), and in the integrals over ∂T, the g-normal $\hat\nu$ points into the element T. With this definition, in view of Lemma 3.3, we have

$$b(g;\sigma,U)=-2\,\widetilde{\operatorname{Inc}\sigma}(\mathbb{A}U),\qquad U\in\mathring{\mathcal{U}}. \tag{4.24}$$

Moreover, if $A = \Phi$ for some $\Phi \in \mathcal{DA}$, then by Theorem 4.1, this extended definition coincides with $\widetilde{\operatorname{Inc}\sigma}(\Phi)$ as defined in (4.8).

The remainder of this section focuses on an adjoint-like operator of the above-defined generalized Inc. This is needed later for the numerical analysis. We have already seen in (4.8) that $\widetilde{\operatorname{Inc}\sigma}(\Phi) = \int_\Omega \langle \sigma, \operatorname{div}\operatorname{div} S\Phi\rangle \omega$ for smooth $\Phi \in \mathcal{DA}$. This motivates the next definition.

Definition 4.6. For any $B \in \mathring{\mathcal{B}}$, we define $\widetilde{\operatorname{div}\operatorname{div}}B$ as a linear functional on $\operatorname{Reg}(\mathscr{T})$ by

$$(\widetilde{\operatorname{div}\operatorname{div}}B)(\sigma) := -\widetilde{\operatorname{Inc}\sigma}(SB), \qquad B \in \mathring{\mathcal{B}},\ \sigma \in \operatorname{Reg}(\mathscr{T}),$$

where the right-hand side is as in Definition 4.5.

Note that neither σ nor g are assumed to be globally smooth in the next result. Given $T \in \mathscr{T}$, $F \in \triangle_{-1}T$ and $E \in \triangle_{-1}F$, in analogy with (3.35), and using the oriented g-normals and g-conormals there (see Figure 1), define

$$[\![B_{\hat\nu\hat\mu EE}]\!]_F^E = B^+_{\hat\nu_F^{T_+}\hat\mu_E^F EE} + B^-_{\hat\nu_F^{T_-}\hat\mu_E^F EE}. \tag{4.25}$$

Here, per the notation in (2.3), the subscript E indicates arguments that are projected onto the tangent space of E. The proof of the next theorem is given after a few necessary lemmas.

Theorem 4.7. *Let $\sigma \in \operatorname{Reg}(\mathscr{T})$, $g \in \operatorname{Reg}^+(\mathscr{T})$, and $B \in \mathring{\mathcal{B}}$. Then*

$$\begin{aligned}(\widetilde{\operatorname{div}\operatorname{div}}B)(\sigma) = \sum_{T\in\mathscr{T}} \Big[\int_T \langle \sigma, \operatorname{div}\operatorname{div}B\rangle \omega_T + \int_{\partial T} \Big(\langle \sigma_F, (\operatorname{div}B + \operatorname{div}^F B)_{\hat\nu\cdot\cdot}\rangle \\ + \sigma_F : \overline{\mathrm{I\!I}}^{\hat\nu} : B_{F\hat\nu\hat\nu F} - \langle \mathrm{I\!I}^{\hat\nu} \otimes \sigma_F, B\rangle \Big) \omega_{\partial T} \Big] \\ + \sum_{E\in\mathring{\mathscr{E}}} \sum_{F\in\mathscr{F}_E} \int_E \langle \sigma_E, [\![B_{\hat\nu\hat\mu EE}]\!]_F^E\rangle \omega_E,\end{aligned} \tag{4.26}$$

where the triple product $\sigma_F : \overline{\mathrm{I\!I}}^{\hat\nu} : B_{\hat\nu\hat\nu} := (\sigma_F)_{ij} \overline{\mathrm{I\!I}}^{\hat\nu jk} B_{\hat\nu\hat\nu}{}^i{}_k$ is in accordance with (3.31), *$\overline{\mathrm{I\!I}}^{\hat\nu} = \mathbb{S}_F(\mathrm{I\!I}^{\hat\nu}) = \mathrm{I\!I}^{\hat\nu} - H^{\hat\nu} g_F$ the trace-reversed second fundamental form (with mean curvature $H^{\hat\nu}$), the jump $[\![B_{\hat\nu\hat\mu EE}]\!]_F^E$ is as defined in* (4.25), *and in the integrals over ∂T, the g-normal $\hat\nu$ points into the element T.*

Remark 4.8. Note, that in (4.26) only the tt-components of σ are involved in the codimension 1 and 2 terms, which are single-valued for $\sigma \in \operatorname{Reg}(\mathscr{T})$.

Lemma 4.9. *Under the assumptions of Theorem 4.7 there holds*

$$\begin{aligned}\sum_{E\in\mathring{\mathscr{E}}} \sum_{F\supset E} \int_E [\![\langle \sigma, B_{\hat\nu\hat\mu} + 2B_{\hat\nu\hat\nu\hat\mu} \otimes \hat\nu\rangle]\!]_F \,\omega_E \\ = \sum_{E\in\mathring{\mathscr{E}}} \sum_{F\supset E} \int_E (\langle \sigma_E, [\![B_{\hat\nu\hat\mu EE}]\!]_F^E\rangle - [\![\sigma_{\hat\mu\hat\nu}]\!]_F^E B_{\hat\mu\hat\nu\hat\nu\hat\mu})\,\omega_E.\end{aligned}$$

Proof. We split the terms into normal, conormal, and codimension 2 components and only write the non-vanishing ones and simplify

$$\begin{aligned}\langle\sigma, B_{\hat\nu\hat\mu}+2B_{\hat\nu\hat\nu\hat\mu}\otimes\hat\nu^\flat\rangle &= \langle\sigma_E, B_{\hat\nu\hat\mu EE}\rangle+\langle\sigma_{E\hat\nu}, B_{\hat\nu\hat\mu E\hat\nu}+2B_{\hat\nu\hat\nu\hat\mu E}\rangle+\langle\sigma_{\hat\mu E}, B_{\hat\nu\hat\mu\hat\mu E}\rangle\\ &\quad+\sigma_{\hat\mu\hat\nu}(B_{\hat\nu\hat\mu\hat\mu\hat\nu}+2B_{\hat\nu\hat\nu\hat\mu\hat\mu})\\ &=\langle\sigma_E, B_{\hat\nu\hat\mu EE}\rangle-\langle\sigma_{E\hat\nu}, B_{\hat\nu\hat\mu E\hat\nu}\rangle+\langle\sigma_{\hat\mu E}, B_{\hat\nu\hat\mu\hat\mu E}\rangle-\sigma_{\hat\mu\hat\nu}B_{\hat\mu\hat\nu\hat\nu\hat\mu}\\ &=\langle\sigma_E, B_{\hat\nu\hat\mu EE}\rangle-\langle\sigma_{\hat\nu E}, B_{\hat\nu\hat\mu E\hat\nu}\rangle-\langle\sigma_{\hat\mu E}, B_{\hat\mu\hat\mu E\hat\nu}\rangle-\sigma_{\hat\mu\hat\nu}B_{\hat\mu\hat\nu\hat\nu\hat\mu}.\end{aligned}$$

For the second identity we used the (skew-)symmetry properties (4.2) of B, $B_{\hat\nu\hat\nu\hat\mu E}=-B_{\hat\nu\hat\mu E\hat\nu}$ and $B_{\hat\nu\hat\nu\hat\mu\hat\mu}=-B_{\hat\nu\hat\mu\hat\mu\hat\nu}$. For the last equality we used the symmetry of σ and that again by (4.2) $B_{\hat\nu\hat\mu\hat\mu E}=-B_{\hat\mu\hat\mu E\hat\nu}$.

The first and last terms together read due to the continuity conditions on σ and B with the jumps (3.35), cf. Figure 1,

$$\begin{aligned}&\sum_{E\in\mathring{\mathscr{E}}}\sum_{F\supset E}\int_E[\![\langle\sigma_E, B_{\hat\nu\hat\mu EE}\rangle-\sigma_{\hat\mu\hat\nu}B_{\hat\mu\hat\nu\hat\nu\hat\mu}]\!]_F\,\omega_E\\ &\qquad=\sum_{E\in\mathring{\mathscr{E}}}\sum_{F\supset E}\int_E\big(\langle\sigma_E,[\![B_{\hat\nu\hat\mu EE}]\!]_F^E\rangle-[\![\sigma_{\hat\mu\hat\nu}]\!]_F^E B_{\hat\mu\hat\nu\hat\nu\hat\mu}\big)\,\omega_E.\end{aligned}$$

Next we show with the same notation as in the proof of Lemma 4.3 that the sum over the remaining middle two terms is zero. Therefore, we reorder the sum, consider an integral representation of the difference with the rotation tensor R such that $R(0)\hat\nu_-=\hat\nu_-$ and $R(\theta)\hat\nu_-=\hat\nu_+$, and prove that the integrand is zero. To this end, we define $F(X,Y,Z,W):=R(t)X^\flat\otimes R(t)Y^\flat\otimes\sigma_{(R(t)Z)E}\otimes R(t)W^\flat$ and compute

$$\begin{aligned}&\sum_{E\in\mathring{\mathscr{E}}}\sum_{F\supset E}[\![\langle\sigma_{\hat\nu E}, B_{\hat\nu\hat\mu E\hat\nu}\rangle+\langle\sigma_{\hat\mu E}, B_{\hat\mu\hat\mu E\hat\nu}\rangle]\!]_F\\ &=\sum_{E\in\mathring{\mathscr{E}}}\sum_{T\supset E}\langle\hat\nu_+^\flat\otimes\hat\mu_+^\flat\otimes\sigma_{\hat\nu_+E}\otimes\hat\nu_+^\flat-\hat\nu_-^\flat\otimes\hat\mu_-^\flat\otimes\sigma_{\hat\nu_-E}\otimes\hat\nu_-^\flat\\ &\qquad\qquad+\hat\mu_+^\flat\otimes\hat\mu_+^\flat\otimes\sigma_{\hat\mu_+E}\otimes\hat\nu_+^\flat-\hat\mu_-^\flat\otimes\hat\mu_-^\flat\otimes\sigma_{\hat\mu_-E}\otimes\hat\nu_-^\flat, B\rangle\\ &=\sum_{E\in\mathring{\mathscr{E}}}\sum_{T\supset E}\int_0^\theta\frac{d}{dt}\langle F(\hat\nu_-,\hat\mu_-,\hat\nu_-,\hat\nu_-)+F(\hat\mu_-,\hat\mu_-,\hat\mu_-,\hat\nu_-), B\rangle\,dt\\ &=\sum_{E\in\mathring{\mathscr{E}}}\sum_{T\supset E}\int_0^\theta\langle F(\hat\mu_-,\hat\mu_-,\hat\nu_-,\hat\nu_-)-F(\hat\nu_-,\hat\nu_-,\hat\nu_-,\hat\nu_-)+F(\hat\nu_-,\hat\mu_-,\hat\mu_-,\hat\nu_-)\\ &\qquad\qquad+F(\hat\nu_-,\hat\mu_-,\hat\nu_-,\hat\mu_-)-F(\hat\nu_-,\hat\mu_-,\hat\mu_-,\hat\nu_-)-F(\hat\mu_-,\hat\nu_-,\hat\mu_-,\hat\nu_-)\\ &\qquad\qquad-F(\hat\mu_-,\hat\mu_-,\hat\nu_-,\hat\nu_-)+F(\hat\mu_-,\hat\mu_-,\hat\mu_-,\hat\mu_-), B\rangle\,dt\\ &=\sum_{E\in\mathring{\mathscr{E}}}\sum_{T\supset E}\int_0^\theta\langle F(\hat\mu_-,\hat\mu_-,\hat\nu_-,\hat\nu_-)+F(\hat\nu_-,\hat\mu_-,\hat\mu_-,\hat\nu_-)-F(\hat\nu_-,\hat\mu_-,\hat\mu_-,\hat\nu_-)\\ &\qquad\qquad-F(\hat\mu_-,\hat\mu_-,\hat\nu_-,\hat\nu_-), B\rangle\,dt=0.\end{aligned}$$

In the penultimate equation we used the skew symmetries (4.2) of B and recognize in the final step that the remaining terms cancel. This concludes the proof. □

Lemma 4.10. *Under the assumptions of Theorem 4.7 there hold the identities*

$$\langle\sigma, \operatorname{div}^F B_{\hat\nu\cdots}\rangle = \langle\sigma, (\operatorname{div}^F B)_{\hat\nu\cdot\cdot}\rangle - \langle\mathbb{I}^{\hat\nu}\otimes\sigma, B\rangle,$$
$$\langle\sigma, \operatorname{div}^F(B_{\cdot\hat\nu\hat\nu\cdot}\otimes\hat\nu^\flat)\rangle = \langle\sigma_{\hat\nu\cdot}, (\operatorname{div}^F B)_{\hat\nu\hat\nu\cdot}\rangle + \langle\mathbb{I}^{\hat\nu}\otimes\sigma_{\hat\nu\cdot}, B_{\hat\nu\cdots}\rangle - \sigma : \mathbb{I}^{\hat\nu} : B_{\cdot\hat\nu\hat\nu\cdot}\,.$$

Proof. We take the surface divergence (A.17) of the above expression using a g-orthonormal basis $\{\hat\tau_i\}_{i=1}^{N-1}$ of the tangent space. Then we use the Leibnitz rule and that the second fundamental form can be expressed as $\nabla_{\hat\tau_i}\hat\nu^\flat = -\sum_{j=1}^{N-1}\mathbb{I}^{\hat\nu}(\hat\tau_i,\hat\tau_j)\hat\tau_j^\flat$

$$\begin{aligned}\langle\sigma, \operatorname{div}^F B_{\hat\nu\cdots}\rangle &= \sum_{i=1}^{N-1}\langle\sigma, (\nabla_{\hat\tau_i}B)_{\hat\nu\hat\tau_i\cdot\cdot} + B_{(\nabla_{\hat\tau_i}\hat\nu)\hat\tau_i\cdot\cdot}\rangle\\ &= \sum_{i=1}^{N-1}\left(\langle\sigma, (\nabla_{\hat\tau_i}B)_{\hat\tau_i\hat\nu\cdot\cdot}\rangle - \sum_{j=1}^{N-1}\langle\sigma, \mathbb{I}^{\hat\nu}(\hat\tau_i,\hat\tau_j)B_{\hat\tau_j\hat\tau_i\cdot\cdot}\rangle\right)\\ &= \langle\sigma, (\operatorname{div}^F B)_{\hat\nu\cdot\cdot}\rangle - \langle\mathbb{I}^{\hat\nu}\otimes\sigma, B\rangle.\end{aligned}$$

For the second line we used the symmetries (4.2) of B and σ for the first term, and for the last line the symmetry of the second fundamental form.

We proceed analogously for the second identity

$$\begin{aligned}&\langle\sigma, \operatorname{div}^F(B_{\cdot\hat\nu\hat\nu\cdot}\otimes\hat\nu^\flat)\rangle\\ &=\langle\sigma, \sum_{i=1}^{N-1}(\nabla_{\hat\tau_i}B)_{\hat\tau_i\hat\nu\hat\nu\cdot}\otimes\hat\nu^\flat + B_{\hat\tau_i(\nabla_{\hat\tau_i}\hat\nu)\hat\nu\cdot}\otimes\hat\nu^\flat + B_{\hat\tau_i\hat\nu(\nabla_{\hat\tau_i}\hat\nu)\cdot}\otimes\hat\nu^\flat + B_{\hat\tau_i\hat\nu\hat\nu\cdot}\otimes\nabla_{\hat\tau_i}\hat\nu^\flat\rangle\\ &=\langle\sigma_{\hat\nu\cdot}, (\operatorname{div}^F B)_{\hat\nu\hat\nu\cdot}\rangle\\ &\quad-\sum_{i,j=1}^{N-1}\langle\sigma, \mathbb{I}^{\hat\nu}(\hat\tau_i,\hat\tau_j)B_{\hat\tau_i\hat\tau_j\hat\nu\cdot}\otimes\hat\nu^\flat + \mathbb{I}^{\hat\nu}(\hat\tau_i,\hat\tau_j)B_{\hat\tau_i\hat\nu\hat\tau_j\cdot}\otimes\hat\nu^\flat + \mathbb{I}^{\hat\nu}(\hat\tau_i,\hat\tau_j)B_{\hat\tau_i\hat\nu\hat\nu\cdot}\otimes\hat\tau_j\rangle.\end{aligned}$$

By the (skew) symmetry of B and the second fundamental form the second term $\sum_{i,j=1}^{N-1}\mathbb{I}^{\hat\nu}(\hat\tau_i,\hat\tau_j)B_{\hat\tau_i\hat\nu\hat\tau_j\cdot}$ vanishes. The remaining terms can be rewritten using the triple product (3.31) as

$$\begin{aligned}-\sum_{i,j=1}^{N-1}&\langle\sigma, \mathbb{I}^{\hat\nu}(\hat\tau_i,\hat\tau_j)B_{\hat\tau_i\hat\tau_j\hat\nu\cdot}\otimes\hat\nu^\flat + \mathbb{I}^{\hat\nu}(\hat\tau_i,\hat\tau_j)B_{\hat\tau_i\hat\nu\hat\nu\cdot}\otimes\hat\tau_j\rangle\\ &= \langle\mathbb{I}^{\hat\nu}\otimes\sigma_{\hat\nu\cdot}, B_{\hat\nu\cdots}\rangle - \sigma : \mathbb{I}^{\hat\nu} : B_{\cdot\hat\nu\hat\nu\cdot},\end{aligned}$$

which concludes the proof. □

Proof of Theorem 4.7. We start by examining the terms arising from integrating the second covariant derivative by parts using (A.16),

(4.27)
$$\begin{aligned}\int_T\langle\nabla^2\sigma, B\rangle\omega_T &= \int_T -\langle\nabla\sigma, \operatorname{div}B\rangle\omega_T - \int_{\partial T}\langle\nabla\sigma, B_{\hat\nu\cdots}\rangle\omega_{\partial T}\\ &= \int_T\langle\sigma, \operatorname{div}\operatorname{div}B\rangle\omega_T + \int_{\partial T}\langle\sigma, (\operatorname{div}B)_{\hat\nu\cdot\cdot}\rangle\omega_{\partial T} - \int_{\partial T}\langle\nabla\sigma, B_{\hat\nu\cdots}\rangle\omega_{\partial T}.\end{aligned}$$

In the last integrand, by supplying either F or $\hat\nu$ in each dot in $B_{\hat\nu\cdots}$, we can split it into normal and tangential components. From the resulting eight terms, eliminating those that are zero by the skew symmetries (4.2) of B, we obtain

$$\langle \nabla\sigma, B_{\hat\nu\cdots}\rangle = \langle (\nabla\sigma)_{FF\hat\nu}, B_{\hat\nu FF\hat\nu}\rangle + \langle (\nabla\sigma)_{\hat\nu FF}, B_{\hat\nu\hat\nu FF}\rangle + \langle (\nabla\sigma)_{FFF}, B_{\hat\nu FFF}\rangle, \tag{4.28}$$

which we then substitute into (4.27).

We use the resulting identity to simplify the terms arising from Definition 4.5. Letting $s_{\mathring{\mathscr{E}}} = \sum_{E\in\mathring{\mathscr{E}}}\sum_{F\in\mathscr{F}_E}\int_E [\![\sigma_{\hat\mu\hat\nu}]\!]_F^E B_{\hat\mu\hat\nu\hat\nu\hat\mu}\,\omega_E$, we then obtain

$$\begin{aligned}
&(\widetilde{\operatorname{divdiv}} B)(\sigma) - s_{\mathring{\mathscr{E}}} = -\widetilde{\operatorname{Inc}}\,\sigma(SB) - s_{\mathring{\mathscr{E}}}\\
&= \sum_{T\in\mathscr{T}}\int_T \langle \nabla^2\sigma, B\rangle\,\omega_T + \int_{\partial T}\langle \sigma_{\hat\nu\hat\nu}\mathrm{I\!I}^{\hat\nu} + (\nabla\sigma)_{F\hat\nu F} + \nabla(\hat\nu\lrcorner\sigma) - (\nabla\sigma)_{\hat\nu FF}, B_{F\hat\nu\hat\nu F}\rangle\,\omega_{\partial T}\\
&= \sum_{T\in\mathscr{T}}\int_T \langle \sigma, \operatorname{divdiv} B\rangle\,\omega_T + \int_{\partial T}\langle \sigma_{\hat\nu\hat\nu}\mathrm{I\!I}^{\hat\nu} + (\nabla\sigma)_{F\hat\nu F} + \nabla(\hat\nu\lrcorner\sigma) - (\nabla\sigma)_{\hat\nu FF}, B_{F\hat\nu\hat\nu F}\rangle\,\omega_{\partial T}\\
&+ \sum_{T\in\mathscr{T}}\int_{\partial T}\Big[\langle \sigma, (\operatorname{div} B)_{\hat\nu\cdot\cdot}\rangle - \langle (\nabla\sigma)_F, B_{\hat\nu FFF}\rangle + \langle (\nabla\sigma)_{\hat\nu FF} - (\nabla\sigma)_{F\hat\nu F}, B_{F\hat\nu\hat\nu F}\rangle\Big]\,\omega_{\partial T},
\end{aligned}$$

where we have used the skew symmetries (4.2) of B and symmetries of σ to rewrite two terms from (4.28) to clarify the impending cancellation of the last two terms above. One of the surviving terms can be rewritten using Lemma 3.17. Although Lemma 3.17 was proved only for $A \in \mathcal{A}$, using the symmetries of B in (4.2) in place of those of A, we find that (3.32) holds also for B, so the same proof there shows that the identity of the lemma holds also for $B \in \mathcal{B}$. Therefore, $\langle \nabla^F(\hat\nu\lrcorner\sigma), B_{F\hat\nu\hat\nu F}\rangle = \langle (\nabla\sigma)_{F\hat\nu F}, B_{F\hat\nu\hat\nu F}\rangle - \mathrm{I\!I} : \sigma_F : B_{F\hat\nu\hat\nu F}$. These observations lead to

$$\begin{aligned}
(\widetilde{\operatorname{divdiv}} B)(\sigma) = s_{\mathring{\mathscr{E}}} &+ \sum_{T\in\mathscr{T}}\int_T \langle \sigma, \operatorname{divdiv} B\rangle\,\omega_T + \int_{\partial T}\langle \sigma, (\operatorname{div} B)_{\hat\nu\cdot\cdot}\rangle\,\omega_{\partial T}\\
&+ \sum_{T\in\mathscr{T}}\int_{\partial T}\Big[\langle \sigma_{\hat\nu\hat\nu}\mathrm{I\!I}^{\hat\nu}, B_{F\hat\nu\hat\nu F}\rangle - \mathrm{I\!I}^{\hat\nu} : \sigma_F : B_{F\hat\nu\hat\nu F} + s_1\Big]\,\omega_{\partial T},
\end{aligned}\tag{4.29}$$

where $s_1 = \langle (\nabla\sigma)_{F\hat\nu F}, B_{F\hat\nu\hat\nu F}\rangle - \langle (\nabla\sigma)_F, B_{\hat\nu FFF}\rangle$.

Next, let us fix a facet $F \in \triangle_{-1}T$ of an element T and examine the two terms whose difference is s_1. The first term equals

$$\langle (\nabla\sigma)_{F\hat\nu F}, B_{F\hat\nu\hat\nu F}\rangle = \langle (\nabla^F\sigma)_{FF\hat\nu}, B_{F\hat\nu\hat\nu F}\rangle = \langle \nabla^F\sigma, B_{\cdot\hat\nu\hat\nu\cdot}\otimes\hat\nu^\flat\rangle.$$

Its second term can be rewritten, noting that arguments indicated by subscript F are tangentially projected using $Q = \mathrm{id} - \hat\nu\otimes\hat\nu^\flat$, and noting that $(\nabla^F\sigma)_{\hat\nu\cdot\cdot} = 0$, as follows:

$$\begin{aligned}
-\langle (\nabla\sigma)_F, B_{\hat\nu FFF}\rangle &= -\langle \nabla^F\sigma, B_{\hat\nu\cdot FF} - \hat\nu^\flat\otimes B_{\hat\nu\hat\nu FF}\rangle = -\langle \nabla^F\sigma, B_{\hat\nu\cdot FF}\rangle\\
&= -\langle \nabla^F\sigma, B_{\hat\nu\cdot\cdot F}\rangle = -\langle \nabla^F\sigma, B_{\hat\nu\cdot\cdot F} - B_{\hat\nu\cdot\cdot\hat\nu}\otimes\hat\nu^\flat\rangle\\
&= \langle \nabla^F\sigma, B_{\cdot\hat\nu\hat\nu\cdot}\otimes\hat\nu^\flat - B_{\hat\nu\cdots}\rangle,
\end{aligned}$$

where we have used (4.2) twice, once in the second line for $B_{\hat\nu\cdot FF} = B_{\hat\nu\cdot\cdot F}$, and again in the last line for $B_{\hat\nu\cdot\cdot\hat\nu} = B_{\cdot\hat\nu\hat\nu\cdot}$. Therefore, s_1 in (4.29) can be expressed using $\nabla^F\sigma$, which then permits the use of the surface integration by parts formula (A.22) on the facet F. Namely,

$$\int_F s_1\,\omega_F = \int_F \langle \nabla^F \sigma, 2B_{\cdot\hat\nu\hat\nu\cdot} \otimes \hat\nu^\flat - B_{\hat\nu\cdots}\rangle \omega_F$$
$$= \int_F \langle \sigma, -\mathrm{div}^F(2B_{\cdot\hat\nu\hat\nu\cdot} \otimes \hat\nu^\flat - B_{\hat\nu\cdots})\rangle \omega_F - \int_F H^{\hat\nu}\langle \hat\nu^\flat \otimes \sigma, 2B_{\cdot\hat\nu\hat\nu\cdot} \otimes \hat\nu^\flat - B_{\hat\nu\cdots}\rangle \omega_F$$
$$- \int_{\partial F} \langle \sigma, 2B_{\hat\mu\hat\nu\hat\nu\cdot} \otimes \hat\nu^\flat - B_{\hat\nu\hat\mu\cdot\cdot}\rangle \omega_{\partial F},$$

where $\hat\mu$ is the inward pointing g-conormal on ∂F. We use Lemma 4.10 to rewrite the first term on the right-hand side above to get

$$\langle \sigma, \mathrm{div}^F(B_{\hat\nu\cdots} - 2B_{\cdot\hat\nu\hat\nu\cdot} \otimes \hat\nu^\flat)\rangle \omega_F = \langle \sigma, (\mathrm{div}^F B)_{\hat\nu\cdot\cdot}\rangle - \langle \mathrm{I\!I}^{\hat\nu} \otimes \sigma, B\rangle$$
$$- 2\langle \sigma_{\hat\nu\cdot}, (\mathrm{div}^F B)_{\hat\nu\hat\nu\cdot}\rangle - 2\langle \mathrm{I\!I}^{\hat\nu} \otimes \sigma_{\hat\nu\cdot}, B_{\hat\nu\cdots}\rangle + 2\sigma : \mathrm{I\!I}^{\hat\nu} : B_{\cdot\hat\nu\hat\nu\cdot}\,.$$

The term multiplying $H^{\hat\nu}$ in the integral of s_1 can be simplified to

$$\langle \hat\nu^\flat \otimes \sigma, 2B_{\cdot\hat\nu\hat\nu\cdot} \otimes \hat\nu^\flat - B_{\hat\nu\cdots}\rangle = \langle \sigma_{\hat\nu\cdot}, 2B_{\hat\nu\hat\nu\hat\nu\cdot}\rangle - \langle \sigma, B_{\hat\nu\hat\nu\cdot\cdot}\rangle = -\langle \sigma_F, B_{\hat\nu\hat\nu\cdot\cdot}\rangle.$$

Hence,

$$(\widetilde{\mathrm{div}\mathrm{div}} B)(\sigma) = \tilde s_{\mathring{\mathscr E}} + \sum_{T\in\mathscr T} \int_T \langle \sigma, \mathrm{div}\mathrm{div} B\rangle \omega_T + \int_{\partial T} \langle \sigma, (\mathrm{div} B)_{\hat\nu\cdot\cdot}\rangle \omega_{\partial T}$$
$$+ \sum_{T\in\mathscr T} \int_{\partial T} \Big[\langle \sigma_{\hat\nu\hat\nu} \mathrm{I\!I}^{\hat\nu}, B_{F\hat\nu\hat\nu F}\rangle - \mathrm{I\!I}^{\hat\nu} : \sigma_F : B_{F\hat\nu\hat\nu F}\Big] \omega_{\partial T}$$
$$+ \sum_{T\in\mathscr T} \int_{\partial T} \Big[\langle \sigma, (\mathrm{div}^F B)_{\hat\nu\cdot\cdot} - 2(\mathrm{div}^F B)_{\hat\nu\hat\nu\cdot} \otimes \hat\nu^\flat\rangle + H^{\hat\nu}\langle \sigma, B_{\hat\nu\hat\nu FF}\rangle\Big] \omega_{\partial T}$$
$$+ \sum_{T\in\mathscr T} \int_{\partial T} \Big[- \langle \mathrm{I\!I}^{\hat\nu} \otimes \sigma, 2B_{\hat\nu\cdots} \otimes \hat\nu^\flat + B\rangle + 2\sigma : \mathrm{I\!I}^{\hat\nu} : B_{\cdot\hat\nu\hat\nu\cdot}\Big] \omega_{\partial T}, \tag{4.30}$$

where $\tilde s_{\mathring{\mathscr E}} = s_{\mathring{\mathscr E}} + \sum_{T\in\mathscr T} \sum_{F\subset\partial T} \int_{\partial F} \langle \sigma, B_{\hat\nu\hat\mu\cdot\cdot} - 2B_{\hat\mu\hat\nu\hat\nu\cdot} \otimes \hat\nu^\flat\rangle \omega_{\partial F}$ collects all integrals over codimension 2 mesh entities.

Let us collect all integrands on the right-hand side of (4.30) involving the second fundamental form into s_2, and all terms involving the divergence operator into s_3, i.e.,

$$s_2 = \langle \sigma_{\hat\nu\hat\nu} \mathrm{I\!I}^{\hat\nu}, B_{F\hat\nu\hat\nu F}\rangle - \langle \mathrm{I\!I}^{\hat\nu} \otimes \sigma, 2B_{\hat\nu\cdots} \otimes \hat\nu^\flat + B\rangle - \mathrm{I\!I}^{\hat\nu} : \sigma_F : B_{F\hat\nu\hat\nu F} + 2\sigma : \mathrm{I\!I}^{\hat\nu} : B_{\cdot\hat\nu\hat\nu\cdot},$$
$$s_3 = \langle \sigma, (\mathrm{div} B)_{\hat\nu\cdot\cdot} + (\mathrm{div}^F B)_{\hat\nu\cdot\cdot}\rangle - \langle \sigma_{\hat\nu\cdot}, 2(\mathrm{div}^F B)_{\hat\nu\hat\nu\cdot}\rangle.$$

To simplify s_2, first note that using (4.3), the last two terms simplify to $\sigma_F : \mathrm{I\!I}^{\hat\nu} : B_{F\hat\nu\hat\nu F}$. Hence,

$$s_2 = \sigma_F : \mathrm{I\!I}^{\hat\nu} : B_{F\hat\nu\hat\nu F} + \langle \sigma_{\hat\nu\hat\nu} \mathrm{I\!I}^{\hat\nu}, B_{F\hat\nu\hat\nu F}\rangle - \langle \mathrm{I\!I}^{\hat\nu} \otimes \sigma, 2B_{\hat\nu\cdots} \otimes \hat\nu^\flat + B\rangle. \tag{4.31}$$

In the last term, splitting σ by (4.23),

$$\langle \mathrm{I\!I}^{\hat\nu} \otimes \sigma,\, 2B_{\hat\nu\cdots} \otimes \hat\nu^\flat + B\rangle = \langle \mathrm{I\!I}^{\hat\nu} \otimes \sigma_{F\hat\nu} \otimes \hat\nu^\flat,\, 2B_{\hat\nu\cdots} \otimes \hat\nu^\flat + B\rangle + \langle \sigma_{\hat\nu\hat\nu} \mathrm{I\!I}^{\hat\nu}, 2B_{\hat\nu\cdot\cdot\hat\nu} + B_{\cdot\cdot\hat\nu\hat\nu}\rangle$$
$$+ \langle \mathrm{I\!I}^{\hat\nu} \otimes \sigma_F,\, 2B_{\hat\nu\cdots} \otimes \hat\nu^\flat + B\rangle + \langle \mathrm{I\!I}^{\hat\nu} \otimes \hat\nu^\flat \otimes \sigma_{\hat\nu F},\, 2B_{\hat\nu\cdots} \otimes \hat\nu^\flat + B\rangle$$
$$= \langle \mathrm{I\!I}^{\hat\nu} \otimes \sigma_F, B\rangle + \langle \mathrm{I\!I}^{\hat\nu} \otimes \sigma_{F\hat\nu},\, 2B_{\hat\nu\cdot\cdot F} + B_{\cdot\cdot F\hat\nu}\rangle + \langle \sigma_{\hat\nu\hat\nu} \mathrm{I\!I}^{\hat\nu}, B_{\hat\nu\cdot\cdot\hat\nu}\rangle - \langle \mathrm{I\!I}^{\hat\nu} \otimes \sigma_{\hat\nu F}, B_{\hat\nu\cdot\cdot F}\rangle$$
$$= \langle \mathrm{I\!I}^{\hat\nu} \otimes \sigma_F, B\rangle + \langle \sigma_{\hat\nu\hat\nu} \mathrm{I\!I}^{\hat\nu}, B_{\hat\nu\cdot\cdot\hat\nu}\rangle$$

by the skew symmetries (4.2) of B and the symmetry of σ. Hence (4.31) simplifies, after using (4.3), to

$$s_2 = \sigma_F : \mathrm{II}^{\hat\nu} : B_{F\hat\nu\hat\nu F} - \langle \mathrm{II}^{\hat\nu} \otimes \sigma_F, B\rangle. \tag{4.32}$$

To simplify s_3, splitting σ by (4.23) again, and using the corresponding splits for the divergence terms, the first inner product in s_3 splits into four terms and the second into two terms. Terms with $(\mathrm{div}^F B)_{\hat\nu\cdot\hat\nu}$ and $(\mathrm{div} B)_{\hat\nu\cdot\hat\nu}$ vanish due to the skew symmetries (4.2) of B. What remains gives

$$s_3 = \langle \sigma_F, (\mathrm{div} B + \mathrm{div}^F B)_{\hat\nu FF}\rangle + \langle \sigma_{\hat\nu F}, (\mathrm{div} B + \mathrm{div}^F B)_{\hat\nu\hat\nu F}\rangle - 2\langle \sigma_{\hat\nu F}, (\mathrm{div}^F B)_{\hat\nu\hat\nu F}\rangle.$$

Since $(\nabla_X B)_{\hat\nu\hat\nu\hat\nu\cdot}$ and $(\nabla_X B)_{\cdot\hat\nu\hat\nu\hat\nu}$ vanish, it is easy to see from (A.17) that there holds $(\mathrm{div}^F B)_{\hat\nu\hat\nu F} = (\mathrm{div} B)_{\hat\nu\hat\nu\cdot}$. Thus s_3 further simplifies to

$$s_3 = \langle \sigma_F, (\mathrm{div} B)_{\hat\nu FF} + (\mathrm{div}^F B)_{\hat\nu FF}\rangle. \tag{4.33}$$

Finally, we simplify $\tilde s_{\mathring{\mathscr{E}}}$ by Lemma 4.9,

$$\begin{aligned} \tilde s_{\mathring{\mathscr{E}}} &= s_{\mathring{\mathscr{E}}} + \sum_{T\in\mathscr{T}} \sum_{F\subset\partial T} \int_{\partial F} \langle \sigma, B_{\hat\nu\hat\mu\cdot\cdot} - 2B_{\hat\mu\hat\nu\hat\nu\cdot} \otimes \hat\nu^\flat\rangle \omega_{\partial F} \\ &= \sum_{E\in\mathring{\mathscr{E}}} \sum_{F\in\mathscr{F}_E} \int_E \langle \sigma_E, [\![B_{\hat\nu\hat\mu}]\!]_F^E\rangle \omega_E. \end{aligned} \tag{4.34}$$

Using (4.34), (4.33), and (4.32) in (4.30), the theorem is proved. □

5. Numerical analysis

In this section, we prove *a priori* convergence estimates for the densitized distributional Riemann curvature tensor in the following setting. Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, be a domain equipped with a smooth "exact" metric tensor $\bar g$. We assume that a family of shape regular triangulations $\{\mathscr{T}_h\}_{h>0}$ consisting of possibly polynomially curved elements of Ω with meshsize $h := \max_{T\in\mathscr{T}_h} h_T$, $h_T := \mathrm{diam}(T)$, are given together with a family of Regge metrics $\{g_h\}_{h>0}$. They approximate $\bar g$ in a sense made precise shortly. Here, shape regularity means that there exists a constant $C_0 > 0$ independent of h such that for all $h > 0$

$$\sup_{T\in\mathscr{T}_h} \frac{h_T}{\rho_T} \leqslant C_0,$$

where ρ_T is the inradius of T. Our convergence estimates will have constants that may depend on C_0.

In finite element computations, we use a reference element $\hat T$, the unit N-simplex, and the space $\mathcal{P}^k(\hat T)$ of polynomials of degree at most k on $\hat T$. Let $\tilde T$ denote a Euclidean N-simplex with possibly polynomially curved facets that is diffeomorphic to $\hat T$ via $\hat\Phi : \hat T \to \tilde T$, $\hat\Phi \in \mathcal{P}^k(\hat T, \mathbb{R}^N)$. Define the *Regge finite element space* of degree k by

$$\mathrm{Reg}_h^k = \{\sigma \in \mathrm{Reg}(\mathscr{T}_h) : \text{for all } T \in \mathscr{T}_h,\ \sigma|_T = \sigma_{ij} dx^i \otimes dx^j,\ \sigma_{ij} \circ \Phi_T \in \mathcal{P}^k(\hat T)\},$$

where $\mathrm{Reg}(\mathscr{T}_h)$ is as in (2.4).

Remark 5.1. For finite element computations on general manifolds M, we would need charts so that each whole element $T \in \mathscr{T}_h$ of the manifold is covered by a single chart giving the coordinates x^i on T. The chart identifies the parameter domain of T as the (possibly curved) Euclidean N-simplex $\tilde T$ diffeomorphic to T.

Let $\Phi : T \to \tilde{T}$ denote this diffeomorphism. Then $\Phi_T = \hat{\Phi}^{-1} \circ \Phi : T \to \hat{T}$ maps T diffeomorphically to the reference element where $\mathcal{P}^k(\hat{T})$ is defined. We use its pullback Φ_T^* below, which is simply the composition with Φ_T for scalar functions. In the setting we have been using, $M = \Omega$ is an open subset of $\mathbb{R}^N$ coverable by a single chart with Φ set globally to the identity.

Throughout, we use standard Sobolev spaces $W^{s,p}(\Omega)$ and their norms and seminorms for any $s \geqslant 0$ and $p \in [1, \infty]$. When the domain is Ω, we omit it from the norm notation if there is no chance of confusion. We also use the elementwise norms $\|u\|^p_{W_h^{s,p}} = \sum_{T \in \mathscr{T}_h} \|u\|^p_{W^{s,p}(T)}$, with the usual adaption for $p = \infty$. When $p = 2$, we put $\|\cdot\|_{H_h^s} = \|\cdot\|_{W_h^{s,2}}$. Furthermore, let

$$\|\sigma\|_2^2 = \|\sigma\|_{L^2}^2 + h^2 \|\sigma\|^2_{H_h^1}, \qquad \|\!|\sigma|\!\|_2^2 = \|\sigma\|_{L^2}^2 + h^2\|\sigma\|^2_{H_h^1} + h^4 \|\sigma\|^2_{H_h^2}.$$

We use $a \lesssim b$ to indicate that there is an h-independent generic constant $C > 0$, depending on Ω and the shape-regularity C_0 of the mesh $\mathscr{T}_h$, such that $a \leqslant Cb$. The C may additionally depend on $\{\|\bar{g}\|_{W^{2,\infty}}, \|\bar{g}^{-1}\|_{L^\infty}, N\}$, if not stated otherwise.

5.1. **Statements of the convergence results.** In (3.5), the generalized densitized curvature operator $\widetilde{\mathcal{Q}\omega}$ was defined as a linear functional on the metric-independent mesh-dependent test space $\mathcal{U}$ of (3.1). To highlight the dependence on the mesh $\mathscr{T}_h$ and the metric g_h on it, we now refer to $\mathcal{U}$ as $\mathcal{U}(\mathscr{T}_h)$ and to $\widetilde{\mathcal{Q}\omega}$ as $\widetilde{\mathcal{Q}\omega}(g_h)$. Obviously, for the smooth metric $\bar{g}$, we have $\widetilde{\mathcal{Q}\omega}(\bar{g}) = (\mathcal{Q}\,\omega)(\bar{g})$ with the exact smooth curvature $\mathcal{Q}$ defined in (3.8). We prove convergence of $\widetilde{\mathcal{Q}\omega}(g_h)$ to the exact densitized curvature $(\mathcal{Q}\,\omega)(\bar{g})$ in the H^{-2}-norm. For this, we only need the action of $\widetilde{\mathcal{Q}\omega}(g_h)$ on a smoother H^2-subspace of the metric-independent test space $\mathcal{U}(\mathscr{T}_h)$. This subspace, after identifying $\wedge^{N-2}(\Omega)^{\odot 2}$ with symmetric $\tilde{N} \times \tilde{N}$ matrix fields, where

$$\tilde{N} = \binom{N}{N-2} = \frac{N(N-1)}{2},$$

is denoted by $H_0^2(\Omega, \mathbb{R}^{\tilde{N}\times\tilde{N}}_{\mathrm{sym}}) = \{u : \Omega \to \mathbb{R}^{\tilde{N}\times\tilde{N}} \;:\; u_{ij} = u_{ji} \in H_0^2(\Omega)\}$. Define

$$\|f\|_{H^{-2}} := \|f\|_{H^{-2}(\Omega, \mathbb{R}^{\tilde{N}\times\tilde{N}}_{\mathrm{sym}})} = \sup_{U \in H_0^2(\Omega, \mathbb{R}^{\tilde{N}\times\tilde{N}}_{\mathrm{sym}})} \frac{f(U)}{\|U\|_{H^2(\Omega, \mathbb{R}^{\tilde{N}\times\tilde{N}}_{\mathrm{sym}})}}.$$

Theorem 5.2. *Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, be a domain equipped with a smooth Riemannian metric $\bar{g}$. Assume that $\{g_h\}_{h>0}$ is a family of Regge metrics on a shape regular family of triangulations $\{\mathscr{T}_h\}_{h>0}$ of Ω with $\lim_{h\to 0} \|g_h - \bar{g}\|_{L^\infty} = 0$ and $C_1 := \sup_{h>0} \max_{T\in\mathscr{T}_h} \|g_h\|_{W^{2,\infty}(T)} < \infty$. Let*

$$C_g = \begin{cases} 1 + \max_{T\in\mathscr{T}_h}(h_T^{-1}\|\bar{g} - g_h\|_{L^\infty(T)}) + \|\bar{g} - g_h\|_{W_h^{1,\infty}}, & N = 2, \\ 1 + \max_{T\in\mathscr{T}_h}(h_T^{-2}\|\bar{g} - g_h\|_{L^\infty(T)}) + \max_{T\in\mathscr{T}_h}(h_T^{-1}\|\bar{g} - g_h\|_{W^{1,\infty}(T)}), & N \geqslant 3. \end{cases}$$

Then there exists $h_0 > 0$ and a $C > 0$ depending on N, C_0, C_1, $\|\bar{g}\|_{W^{2,\infty}(\Omega)}$, and $\|\bar{g}^{-1}\|_{L^\infty(\Omega)}$ such that for all $h \leqslant h_0$,

$$\|\widetilde{\mathcal{Q}\omega}(g_h) - (\mathcal{Q}\,\omega)(\bar{g})\|_{H^{-2}} \leqslant C C_g \,\|\!|g_h - \bar{g}|\!\|_2\,.$$

When metric approximation is sufficiently good so as to have C_g bounded independently of meshsize, then rates of convergence can be quantified from the above result. As an example, let $\mathcal{I}_h^k : C^\infty(\Omega, \mathcal{S}) \to \mathrm{Reg}_h^k$ be an interpolation operator into the Regge finite element space satisfying the following conditions: there exists a $p \in [2, \infty]$ such that the interpolant can be continuously extended to symmetric tensor fields in $W^{k+1,p}(\Omega)$ and

$$|\mathcal{I}_h^k g - g|_{W^{t,p}(T)} \leqslant C_2 h_T^{k+1-t} |g|_{W^{k+1,p}(T)}, \qquad \text{for all } t \in [0, k+1], \tag{5.1a}$$

where $C_2 = C_2(N, k, h_T/\rho_T, t)$. Additionally suppose that for all $0 < s \leqslant \min\{k+1, 2\}$,

$$|\mathcal{I}_h^k g - g|_{W^{t,\infty}(T)} \leqslant C_2 h_T^{s-t} |g|_{W^{s,\infty}(T)}, \qquad \text{for all } t \in [0, s]. \tag{5.1b}$$

The canonical Regge interpolant [29] is an example of an interpolant fulfilling (5.1). It is well defined for g in $W^{s,p}(\Omega)$ with $s > (N-1)/p$, a sufficient condition for defining traces on boundaries of codimension up to $N-1$. Then setting e.g. $p = N$ meets the requirements. An Oswald interpolant, performing local elementwise L^2-projections and averaging the degrees of freedom shared by different elements, also leads to a valid choice if the requirements are weakened to hold on element patches, see e.g. [21, Appendix A] for an example.

Corollary 5.3. *Suppose the assumptions of Theorem 5.2 hold. Assume further that $g_h = \mathcal{I}_h^k g \in \mathrm{Reg}_h^k$, for some integer $k \geqslant 0$ for $N = 2$ and $k \geqslant 1$ for $N \geqslant 3$, and $\mathcal{I}_h^k$ is an interpolation operator fulfilling (5.1) for some $p \in [2, \infty]$. Then there exists $h_0 > 0$ and $C > 0$ depending on N, Ω, k, C_0, C_1, C_2, $\|\bar{g}\|_{W^{2,\infty}(\Omega)}$, and $\|\bar{g}^{-1}\|_{L^\infty(\Omega)}$ such that for all $h \leqslant h_0$*

$$\|\widetilde{\mathcal{Q}\omega}(g_h) - (\mathcal{Q}\,\omega)(\bar{g})\|_{H^{-2}} \leqslant C \Big(\sum_{T \in \mathscr{T}_h} h_T^{p(k+1)} |\bar{g}|_{W^{k+1,p}(T)} \Big)^{1/p}. \tag{5.2}$$

For $p = \infty$ the right-hand side is $C \max_{T \in \mathscr{T}_h} h_T^{k+1} |g|_{W^{k+1,\infty}(T)}$.

Remark 5.4 (Lowest order cases). Note that in the case $N = 2$, Corollary 5.3 gives convergence already for the lowest-order case of piecewise constant Regge metrics. In $N \geqslant 3$ dimensions however, at least linear elements are needed to obtain norm convergence from (5.2). In §7 we will confirm this requirement through a numerical example in $N = 3$ case. It is consistent with the results in [21, Theorem 4.1] and [22, Corollary 4.3] where, for dimensions greater than two, at least linear Regge elements were needed to obtain convergence for the densitized distributional scalar curvature and Einstein tensor, respectively.

Remark 5.5 (Applicability to embedded hypersurfaces). Let M_h be a piecewise smooth N-dimensional hypersurface embedded in $\mathbb{R}^{N+1}$ approximating a smooth hypersurface. Let $\mathcal{V}_h^k$ denote the degree k Lagrange finite element space on a mesh $\mathscr{T}_h$ of Ω, a subspace of the space in (2.16). Letting $\Omega \subset \mathbb{R}^N$ take the role of a parameter domain for the hypersurface, assume that $M_h = \Phi_h(\Omega)$ for some embedding $\Phi_h \in [\mathcal{V}_h^k]^{N+1}$. Then, the induced metric tensor $g_h = (\operatorname{grad} \Phi_h)^T \operatorname{grad} \Phi_h$ is tt-continuous, and thus defines a Regge metric in $\mathrm{Reg}_h^{2(k-1)}$. Therefore, Theorem 5.2 and Corollary 5.3 can be applied.

5.2. **Roadmap of the proof.** To prove Theorem 5.2 and Corollary 5.3 we use (3.36) to rewrite the error as an integral representation. Namely, letting $g(t) := \bar{g} + (g_h - \bar{g})t$ and $\sigma := g'(t) = g_h - \bar{g}$, by the fundamental theorem of calculus and

Theorem 3.20, we obtain the following integral representation of the error

$$\Big(\widetilde{\mathfrak{Q}\omega}(g_h) - (\mathfrak{Q}\,\omega)(\bar{g})\Big)(U) = \int_0^1 \frac{d}{dt}\widetilde{\mathfrak{Q}\omega}(g(t))(U)\,dt$$
$$= \int_0^1 \left(a(g(t);\sigma,U) + b(g(t);\sigma,U)\right)\,dt,$$

where $a(\cdot;\cdot,\cdot)$ and $b(\cdot;\cdot,\cdot)$ are defined as in (3.37) and (3.38), respectively.

We estimate bilinear form $a(\cdot;\cdot,\cdot)$ directly in Proposition 5.12 below, whereas for bilinear form $b(\cdot;\cdot,\cdot)$ we use the relation of the distributional covariant incompatibility operator, (4.24), and estimate it via its adjoint, (4.26) of Theorem 4.7,

$$b(g(t);\sigma,U) = -2\,\widetilde{\operatorname{Inc}\sigma}(B) = 2\,(\widetilde{\operatorname{divdiv}B})(\sigma),$$

with $B = S\mathbb{A}_{g(t)}(U)$, as done in Proposition 5.17 below.

5.3. **Basic estimates.** We assume that the approximation property $\lim_{h\to 0}\|g_h - \bar{g}\|_{L^\infty(\Omega)} = 0$ and stability estimate $\sup_{h>0}\max_{T\in\mathscr{T}_h}\|g_h\|_{W^{2,\infty}(T)} < \infty$, both assumptions of Theorem 5.2, tacitly hold in the remainder of this section. These assumptions have some elementary consequences that we record here for reference (see e.g. [20] for a derivation). For every h sufficiently small, every $t \in [0,1]$, and every vector w with unit Euclidean length there holds

$$\|g(t)\|_{L^\infty} + \|g(t)^{-1}\|_{L^\infty} \lesssim 1, \qquad \max_{T\in\mathscr{T}_h}|g(t)|_{W^{2,\infty}(T)} \lesssim 1, \tag{5.3a}$$

$$1 \lesssim \inf_{\Omega}(w^T g(t) w) \leqslant \sup_{\Omega}(w^T g(t) w) \lesssim 1, \tag{5.3b}$$

where we interpret $g(t)$ as matrix and w as column vector in (5.3b). Note that (5.3b) implies the existence of positive lower and upper bounds on the inverse as well

$$1 \lesssim \inf_{\Omega}(w^T g(t)^{-1} w) \leqslant \sup_{\Omega}(w^T g^{-1}(t) w) \lesssim 1.$$

In addition, the inequalities $\|g(t)\|_{L^\infty} \lesssim 1$ and $\|g^{-1}(t)\|_{L^\infty} \lesssim 1$ imply that

$$\begin{aligned}\|\rho\|_{L^p(D,g(t_2))} &\lesssim \|\rho\|_{L^p(D,g(t_1))} \lesssim \|\rho\|_{L^p(D,g(t_2))},\\ \|\rho\|_{L^p(D)} &\lesssim \|\rho\|_{L^p(D,g(t_1))} \lesssim \|\rho\|_{L^p(D)}\end{aligned} \tag{5.3c}$$

for every $t_1, t_2 \in [0,1]$, every admissible submanifold D, every $p \in [1,\infty]$, every tensor field ρ having finite $L^p(D)$-norm, and every h sufficiently small. We select $h_0 > 0$ so that (5.3a–5.3c) hold for all $h \leqslant h_0$, and we tacitly use these inequalities throughout our analysis.

We will also need the following auxiliary result.

Lemma 5.6. *Let g_1 and g_2 be two symmetric positive definite matrices, and let ν be a Euclidean unit vector. Let with the notation $g_i^{\nu\nu} = g_i^{ab}\nu_a\nu_b$*

$$\hat{\nu}_{g_i} = \frac{1}{\sqrt{g_i^{\nu\nu}}}g_i^{-1}\nu, \quad i = 1,2.$$

Then there exists a constant $c > 0$ depending on the Euclidean norms $|g_1|$, $|g_2|$, $|g_1^{-1}|$, $|g_2^{-1}|$ such that

$$|\hat{\nu}_{g_1} - \hat{\nu}_{g_2}| \leqslant c|g_1 - g_2|.$$

Proof. See e.g. [21, Lemma 4.6]. □

As preparation, we estimate some geometric quantities arising frequently.

Lemma 5.7. *Let $D \in \{T, F, E\}$ be a volume, codimension 1, or codimension 2 domain of $\mathscr{T}_h$, $\mathbb{A}_{g(t)}$ the mapping defined in* (3.2)*, and $g(t) = \bar{g} + (g_h - \bar{g})t$. There holds for all $t \in [0,1]$ and $p \in [1,\infty]$*

$$\begin{aligned}\|\mathbb{A}_{g(t)}(U)\|_{L^p(D)} &\lesssim \|U\|_{L^p(D)},\\ \|\mathbb{A}_{g(t)}(U) - \mathbb{A}_{\bar{g}}(U)\|_{L^p(D)} &\lesssim \|\bar{g} - g_h\|_{L^\infty(D)}\|U\|_{L^p(D)},\\ \|\omega_D(g(t))\|_{L^\infty(D)} + \|\mathcal{R}_{g(t)}\|_{L^\infty(T)} + \|\hat{\nu}_{g(t)}\|_{L^\infty(F)} &\lesssim 1.\end{aligned}$$

Proof. Follows directly by the assumptions on $\bar{g}$ and g_h and the definition (3.2) of $\mathbb{A}_g$ with coordinate expression (3.9). □

The following estimates on boundary facets are crucial for the analysis.

Lemma 5.8. *Let $U \in H_0^2(\Omega, \mathbb{R}^{\tilde{N}\times\tilde{N}}_{\mathrm{sym}})$, $g(t) = \bar{g} + (g_h - \bar{g})t$, and $A_{g(t)} = \mathbb{A}_{g(t)}U$. There holds for all $F \in \mathring{\mathscr{F}}_h$ and $t \in [0,1]$*

$$\begin{aligned}\|[\![\mathrm{I\!I}_{g(t)}]\!]\|_{L^\infty(F)} &\lesssim \|[\![g_h - \bar{g}]\!]\|_{W^{1,\infty}(F)},\\ \|[\![(\mathrm{div}_{g(t)} A_{g(t)} + \mathrm{div}^F_{g(t)} S A_{g(t)})_{\hat{\nu}_{g(t)}}]\!]\|_{L^2(F)} &\lesssim \|[\![\bar{g} - g_h]\!]\|_{W^{1,\infty}(F)}\|U\|_{L^2(F)}\\ &\quad + \|[\![g_h - \bar{g}]\!]\|_{L^\infty(F)}\,|U|_{H^1(F)}.\end{aligned}$$

If g_h is piecewise constant the $W^{1,\infty}(F)$-norm in both inequalities can be replaced by the $L^\infty(F)$-norm.

Proof. For the first statement see e.g. [21, Lemma 4.9].

To prove the second claim we first define the abbreviation $\tilde{g} := g(t)$ and consider the covariant divergence of a fourth order tensor in coordinates with Γ_{ij}^k denoting the Christoffel symbols of second kind with respect to $\tilde{g}$

$$\begin{aligned}(\mathrm{div}_{\tilde{g}} A_{\tilde{g}})_{jkl} = g^{rs}\partial_r (A_{\tilde{g}})_{sjkl} &- g^{rs}\Gamma^p_{rs}(A_{\tilde{g}})_{pjkl} - g^{rs}\Gamma^p_{rj}(A_{\tilde{g}})_{spkl}\\ &- g^{rs}\Gamma^p_{rk}(A_{\tilde{g}})_{sjpl} - g^{rs}\Gamma^p_{rl}(A_{\tilde{g}})_{sjkp}.\end{aligned}$$

Using (iteratively) that $[\![ab]\!] = [\![a]\!]\{\{b\}\} + \{\{a\}\}[\![b]\!]$, with $\{\{a\}\} := 0.5(a^+ + a^-)$ the mean value of a, coordinate expression (3.9) yields for the first term in the divergence expression for each k, l

$$\begin{aligned}&\|[\![g^{rs}\partial_r(A_{\tilde{g}})_{sjkl}\hat{\nu}^j_{\tilde{g}}]\!]\|_{L^2(F)} \lesssim \|[\![\partial_i(A_{\tilde{g}})^{ijkl}(\hat{\nu}_{\tilde{g}})_j]\!]\|_{L^2(F)}\\ &\lesssim \|[\![(\partial_i \det\tilde{g}^{-1})\varepsilon^{ij\,\alpha}\varepsilon^{kl\,\beta}U_{\alpha\beta} + \det\tilde{g}^{-1}\varepsilon^{ij\,\alpha}\varepsilon^{kl\,\beta}\partial_i U_{\alpha\beta}]\!]\|_{L^2(F)}\|\{\{\hat{\nu}_{\tilde{g}}\}\}\|_{L^\infty(F)}\\ &\quad + \|[\![\hat{\nu}_{\tilde{g}}]\!]\|_{L^\infty(F)}\|\{\{(\partial_i \det\tilde{g}^{-1})\varepsilon^{ij\,\alpha}\varepsilon^{kl\,\beta}U_{\alpha\beta} + \det\tilde{g}^{-1}\varepsilon^{ij\,\alpha}\varepsilon^{kl\,\beta}\partial_i U_{\alpha\beta}\}\}\|_{L^2(F)}\\ &\lesssim |[\![\tilde{g}]\!]|_{W^{1,\infty}(F)}\|U\|_{L^2(F)} + \|[\![\tilde{g}]\!]\|_{L^\infty(F)}|U|_{H^1(F)} + \|[\![\hat{\nu}_{\tilde{g}}]\!]\|_{L^\infty(F)}(\|U\|_{L^2(F)} + |U|_{H^1(F)}).\end{aligned}$$

For the last inequality we used Lemma 5.6 and Lemma 5.7. Using that the jump is zero for the exact smooth metric $\bar{g}$ together with $\tilde{g} - \bar{g} = t(g_h - \bar{g})$ and Lemma 5.6

$$\begin{aligned}\|[\![\tilde{g}]\!]\|_{W^{1,\infty}(F)} &= \|[\![\tilde{g} - \bar{g}]\!]\|_{W^{1,\infty}(F)} \lesssim \|[\![g_h - \bar{g}]\!]\|_{W^{1,\infty}(F)},\\ \|[\![\hat{\nu}_{\tilde{g}}]\!]\|_{L^\infty(F)} &= \|[\![\hat{\nu}_{\tilde{g}} - \hat{\nu}_{\bar{g}}]\!]\|_{L^\infty(F)} \lesssim \|[\![g_h - \bar{g}]\!]\|_{L^\infty(F)},\end{aligned}$$

we obtain the estimate

$$\|g^{rs}[\![\partial_r(A_{\tilde{g}})_{sjkl}\hat{\nu}^j_{\tilde{g}}]\!]\|_{L^2(F)} \lesssim \|[\![g_h - \bar{g}]\!]\|_{L^\infty(F)}\,|U|_{H^1(F)} + \|[\![g_h - \bar{g}]\!]\|_{W^{1,\infty}(F)}\|U\|_{L^2(F)}.$$

Analogously there holds for the remaining three terms of the covariant divergence $(\operatorname{div}_{\tilde g} A_{\tilde g})_{jkl}$

$$\begin{aligned}
&\|[\![g^{rs}\Gamma^p_{rs}(A_{\tilde g})_{pjkl}\hat\nu^j_{\tilde g}]\!]\|_{L^2(F)} \\
&\qquad \leqslant \|[\![\Gamma]\!]\|_{L^\infty(F)}\|\{\!\{(A_{\tilde g})_{ijkl}\hat\nu^j_{\tilde g}\}\!\}\|_{L^2(F)} + \|\{\!\{\Gamma\}\!\}\|_{L^\infty(F)}\|[\![(A_{\tilde g})_{ijkl}\hat\nu^j_{\tilde g}]\!]\|_{L^2(F)} \\
&\qquad \lesssim \|[\![\bar g - g_h]\!]\|_{W^{1,\infty}(F)}\|U\|_{L^2(F)} + \|[\![\bar g - g_h]\!]\|_{L^\infty(F)}\|U\|_{L^2(F)}.
\end{aligned}$$

As the surface divergence $\operatorname{div}^F_{g(t)} SA_{g(t)}$ can be estimated in the same way we obtain the second claim. Finally, we note that for piecewise constant g_h there holds due to the smoothness of $\bar g$ that $\|[\![g_h - \bar g]\!]\|_{W^{1,\infty}(F)} = \|[\![g_h - \bar g]\!]\|_{L^\infty(F)}$. □

5.4. **Estimating bilinear form a.** We start by estimating the terms involved in (3.37). Note that the appearing inner products have to be understood with respect to the metric $g(t) = \bar g + (g_h - \bar g)t$ and recall that $\sigma = g_h - \bar g$.

Lemma 5.9. *There holds for the volume term of* (3.37) *for all* $t \in [0,1]$

$$\Big|\sum_{T\in\mathscr{T}_h}\int_T \big(\langle L^{(1)}_\sigma \mathcal{R}_{g(t)}, A_{g(t)}\rangle - \frac{1}{2}\operatorname{tr}_{g(t)}(\sigma)\langle \mathcal{R}_{g(t)}, A_{g(t)}\rangle\big)\omega_T(g(t))\Big| \lesssim \|\bar g - g_h\|_{L^2}\|U\|_{L^2},$$

where $L^{(1)}_\sigma$, *cf.* (3.15), *has to be understood with respect to* $g(t)$.

Proof. Follows directly by Hölder inequality and Lemma 5.7, e.g.,

$$\begin{aligned}
\Big|\int_T \langle L^{(1)}_\sigma \mathcal{R}_{g(t)}, A_{g(t)}\rangle \omega_T(g(t))\Big| &\lesssim \|\mathcal{R}_{g(t)}\|_{L^\infty(T)}\|\sigma\|_{L^2(T)}\|A_{g(t)}\|_{L^2(T)} \\
&\lesssim \|\sigma\|_{L^2(T)}\|U\|_{L^2(T)}.
\end{aligned}$$

□

Lemma 5.10. *There holds for the codimension 1 term of* (3.37) *for all* $t \in [0,1]$

$$\begin{aligned}
\Bigg|\sum_{F\in\mathring{\mathscr{F}}_h}\int_F [\![\mathrm{II}_{g(t)}]\!] : \mathbb{S}_F(\sigma) : A_{g(t),F\hat\nu_{g(t)}\hat\nu_{g(t)}F}\,\omega_F(g(t))\Bigg| &\lesssim \max_{T\in\mathscr{T}_h}\big(h_T^{-1}\|g_h - \bar g\|_{W^{1,\infty}(T)}\big) \\
&\quad \times \|\!|g_h - \bar g|\!\|_2 \,\|\!|U|\!\|_2\,.
\end{aligned}$$

If g_h is piecewise constant the $W^{1,\infty}(T)$-norm can be replaced by the $L^\infty(T)$-norm.

Proof. Using Lemma 5.7 and Lemma 5.8 there holds noting that $\|\mathbb{S}_F(\sigma)\|_{L^2(F)} \lesssim \|\sigma_F\|_{L^2(F)}$

$$\begin{aligned}
\Big|\int_F [\![\mathrm{II}_{g(t)}]\!]:\mathbb{S}_F(\sigma):A_{g(t),F\hat\nu_{g(t)}\hat\nu_{g(t)}F}\,\omega_F(g(t))\Big| &\lesssim \|\mathbb{S}_F(\sigma)\|_{L^2(F)}\|\|[\![\mathrm{II}_{g(t)}]\!]\|_{L^\infty(F)}\|U\|_{L^2(F)} \\
&\lesssim \|\sigma_F\|_{L^2(F)}\|[\![g_h - \bar g]\!]\|_{W^{1,\infty}(F)}\|U\|_{L^2(F)}.
\end{aligned}$$

With the trace inequality

$$\|U\|^2_{L^2(F)} \leqslant C(h_T^{-1}\|U\|^2_{L^2(T)} + h_T|U|^2_{H^1(T)}), \qquad F \subset \partial T, \tag{5.4}$$

we get

$$\begin{aligned}
&\|\sigma_F\|^2_{L^2(F)}\|[\![g_h - \bar g]\!]\|^2_{W^{1,\infty}(F)}\|U\|^2_{L^2(F)} \\
&\lesssim \big(h_T^{-1}\sum_{i=1}^2 \|g_h - \bar g\|_{W^{1,\infty}(T_i)}\big)^2(\|\sigma\|^2_{L^2(T_1)} + h^2_{T_1}|\sigma|^2_{H^1(T_1)})(\|U\|^2_{L^2(T_1)} + h^2_{T_1}|U|^2_{H^1(T_1)}).
\end{aligned}$$

Due to the shape-regularity of $\mathscr{T}_h$, we have $C^{-1} \leqslant h_{T_1}/h_{T_2} \leqslant C$ for some constant C independent of h and F, and thus

$$\left|\int_F [\![\mathrm{I\!I}_{g(t)}]\!] : \mathbb{S}_F(\sigma) : A_{g(t),F\hat{\nu}_{g(t)}\hat{\nu}_{g(t)}F}\, \omega_F(g(t))\right| \lesssim \max_{T\in\mathscr{T}_h} \left(h_T^{-1}\|g_h - g\|_{W^{1,\infty}(T)}\right) \times |\!|\!| g_h - \bar{g} |\!|\!|_2 \, |\!|\!| U |\!|\!|_2 \,.$$

If g_h is piecewise constant there holds by the smoothness of the exact metric $\bar{g}$ that $\|[\![g_h - \bar{g}]\!]\|_{W^{1,\infty}(F)} = \|[\![g_h - \bar{g}]\!]\|_{L^\infty(F)}$. □

Lemma 5.11. *There holds for the codimension 2 term of* (3.37) *for all* $t \in [0,1]$

$$\left|\sum_{E\in\mathring{\mathscr{E}}_h} \int_E \mathrm{tr}_{g(t)}(\sigma_E)\Theta_E(g(t))(A_{g(t)})_{\hat{\mu}_{g(t)}\hat{\nu}_{g(t)}\hat{\nu}_{g(t)}\hat{\mu}_{g(t)}}\, \omega_E(g(t))\right| \lesssim \left(\max_{T\in\mathscr{T}_h} h_T^{-2}\|g_h - \bar{g}\|_{L^\infty(T)}\right) |\!|\!| g_h - \bar{g} |\!|\!|_2 \, |\!|\!| U |\!|\!|_2 \,.$$

Proof. Follows analogously to [21, Lemma 4.13]. □

Proposition 5.12. *Let* $g(t) = \bar{g} + (g_h - \bar{g})t$, $\sigma = g_h - \bar{g}$, *and* $U \in H_0^2(\Omega, \mathbb{R}^{\tilde{N}\times\tilde{N}}_{\mathrm{sym}})$. *There holds for all* $t \in [0,1]$

$$|a(g(t);\sigma,U)| \lesssim \left(1 + \max_{T\in\mathscr{T}_h} h_T^{-1}\|g_h - \bar{g}\|_{W^{1,\infty}(T)} + \max_{T\in\mathscr{T}_h} h_T^{-2}\|g_h - \bar{g}\|_{L^\infty(T)}\right) \times |\!|\!| g_h - \bar{g} |\!|\!|_2 \, \|U\|_{H^2}.$$

Assume that $g_h = \mathcal{I}_h^k \bar{g}$ *is an interpolant fulfilling assumptions* (5.1) *with a* $p \in [2,\infty]$. *Then for* $k \geqslant 1$

$$\left|a(g(t);\sigma,U)\right| \lesssim \left(\sum_{T\in\mathscr{T}_h} h_T^{p(k+1)}|\bar{g}|_{W^{k+1,p}(T)}\right)^{1/p} \|U\|_{H^2}.$$

Proof. Combine Lemmas 5.9–5.11 and the approximation assumptions (5.1) on the interpolation operator together with

$$\begin{aligned}
&\|g_h - \bar{g}\|_{L^2(\Omega)} \leqslant |\Omega|^{\frac{1}{2}-\frac{1}{p}}\|g_h - \bar{g}\|_{L^p(\Omega)},\\
&\left(\sum_{T\in\mathscr{T}_h} h_T^2|g_h - \bar{g}|^2_{H^1(T)}\right)^{\frac{1}{2}} \leqslant |\Omega|^{\frac{1}{2}-\frac{1}{p}}\left(\sum_{T\in\mathscr{T}_h} h_T^p|g_h - \bar{g}|^p_{W^{1,p}(\Omega)}\right)^{\frac{1}{p}},\\
&\left(\sum_{T\in\mathscr{T}_h} h_T^4|g_h - \bar{g}|^2_{H^2(T)}\right)^{\frac{1}{2}} \leqslant |\Omega|^{\frac{1}{2}-\frac{1}{p}}\left(\sum_{T\in\mathscr{T}_h} h_T^{2p}|g_h - \bar{g}|^p_{W^{2,p}(\Omega)}\right)^{\frac{1}{p}},
\end{aligned} \tag{5.5}$$

for $p \in [2,\infty)$ and standard modifications for $p = \infty$. □

Remark 5.13. Proposition 5.12 shows optimal convergence rates in the H^{-2}-norm despite the lowest-order case of piecewise constant Regge metrics, $k = 0$. The critical term destroying the convergence is the codimension 2 term from Lemma 5.11. This behavior has been observed also for the scalar curvature and Einstein tensor in [21, 22] for $N \geqslant 3$, where the same (adapted) term yields to suboptimal rates. We verify numerically that Lemma 5.11 and thus Proposition 5.12 are sharp. We show in §6.1 that $a(\cdot;\cdot,\cdot)$ is zero in two dimensions.

5.5. **Estimating bilinear form b.** Next, we estimate the terms involved in (3.38), or, more precisely, its adjoint (4.26) of Theorem 4.7

$$b(g(t);\sigma,U) = -2\,\widetilde{\operatorname{Inc}\sigma}(B) = 2\,(\widetilde{\operatorname{div}\operatorname{div}}B)(\sigma), \qquad B = S\mathbb{A}_{g(t)}(U),$$

starting with the volume terms.

Lemma 5.14. *There holds for the volume term of* (4.26) *for all* $t\in[0,1]$

$$\left|\sum_{T\in\mathscr{T}_h}\int_T \langle\sigma, \operatorname{div}_{g(t)}\operatorname{div}_{g(t)}SA_{g(t)}\rangle\,\omega_T(g(t))\right| \lesssim \|\sigma\|_{L^2(\Omega)}\|U\|_{H^2(\Omega)}.$$

Proof. Follows by Hölder inequality and Lemma 5.7. □

Lemma 5.15. *There holds for the codimension 1 term of* (4.26) *for all* $t\in[0,1]$

$$\begin{aligned}\Big|\sum_{F\in\mathring{\mathscr{F}}_h}\int_F &[\![\langle\sigma_F, (\operatorname{div}_{g(t)}SA_{g(t)} + \operatorname{div}^F_{g(t)}SA_{g(t)})_{\hat\nu_{g(t)}}\rangle \\ &+ \sigma_F : \overline{\mathrm{I\!I}}_{g(t)} : (SA)_{g(t),\hat\nu_{g(t)}\hat\nu_{g(t)}} - \langle\mathrm{I\!I}_{g(t)}\otimes\sigma_F, SA_{g(t)}\rangle]\!]\,\omega_F(g(t))\Big| \qquad (5.6)\\ \lesssim |\!|\!|\sigma|\!|\!|_2\Big(&\max_{T\in\mathscr{T}_h}(h_T^{-1}\|\bar g - g_h\|_{W^{1,\infty}(T)})\,|\!|\!|U|\!|\!|_2 + \max_{T\in\mathscr{T}_h}(h_T^{-1}\|\bar g - g_h\|_{L^\infty(T)})\,|\!|\!|\nabla_\delta U|\!|\!|_2\Big),\end{aligned}$$

where $\nabla_\delta U$ denotes the Euclidean gradient instead of the covariant one. If g_h is piecewise constant the $W^{1,\infty}(T)$-norm can be replaced by the $L^\infty(T)$-norm.

Proof. The last two terms of (5.6) involving the second fundamental form can be estimated as in the proof of Lemma 5.10. For the first term we use Hölder and Lemma 5.8

$$\begin{aligned}&\left|\int_F \langle\sigma_F, [\![(\operatorname{div}_{g(t)}SA_{g(t)} + \operatorname{div}^F_{g(t)}SA_{g(t)})_{\hat\nu_{g(t)}}]\!]\rangle\,\omega_F(g(t))\right| \\ &\qquad\lesssim \|\sigma_F\|_{L^2(F)}\|[\![(\operatorname{div}_{g(t)}SA_{g(t)} + \operatorname{div}^F_{g(t)}SA_{g(t)})_{\hat\nu_{g(t)}}]\!]\|_{L^2(F)} \\ &\qquad\lesssim \|\sigma_F\|_{L^2(F)}\Big(\|[\![\bar g - g_h]\!]\|_{W^{1,\infty}(F)}\|U\|_{L^2(F)} + \|[\![g_h - \bar g]\!]\|_{L^\infty(F)}\,|U|_{H^1(F)}\Big).\end{aligned}$$

Applying trace inequality (5.4) and using the shape regularity, as in the proof of Lemma 5.10 gives the desired result

$$\begin{aligned}&\left|\sum_{F\in\mathring{\mathscr{F}}_h}\int_F \langle\sigma_F, [\![(\operatorname{div}_{g(t)}SA_{g(t)} + \operatorname{div}^F_{g(t)}SA_{g(t)})_{\hat\nu_{g(t)}}]\!]\rangle\,\omega_F(g(t))\right| \\ &\lesssim |\!|\!|\sigma|\!|\!|_2\Big(\max_{T\in\mathscr{T}_h}(h_T^{-1}\|\bar g - g_h\|_{W^{1,\infty}(T)})\,|\!|\!|U|\!|\!|_2 + \max_{T\in\mathscr{T}_h}(h_T^{-1}\|\bar g - g_h\|_{L^\infty(T)})\,|\!|\!|\nabla_\delta U|\!|\!|_2\Big).\end{aligned}$$

□

The codimension 2 term of (4.26) is zero for dimension $N = 2$, $\sigma_E = 0$ for 0-dimensional E. In higher spatial dimensions it has to be considered leading to a lower convergence rate than Lemma 5.14 and Lemma 5.15, comparable to the result of Lemma 5.11 for bilinear form $a(\cdot;\cdot,\cdot)$.

Lemma 5.16. *There holds for the codimension 2 term of* (4.26) *for all* $t \in [0,1]$

$$\left|\sum_{E\in\mathring{\mathscr{E}}_h}\sum_{F\supset E}\int_E \langle\sigma_E, [\![(SA_{g(t)})_{\hat\nu_{g(t)}\hat\mu_{g(t)}}EE]\!]_F^E\rangle\,\omega_E(g(t))\right| \lesssim \max_{T\in\mathscr{T}_h}\left(h_T^{-2}\|g_h-g\|_{L^\infty(T)}\right)|||\sigma|||_2\,|||U|||_2\,.$$

Proof. By the shape regularity of $\mathscr{T}_h$ the number of facets attached to $E\in\mathring{\mathscr{E}}$ is bounded by a constant C independent of h. Using that $\sum_{F\supset E}[\![(SA_{\bar g})_{\hat\nu_{\bar g}\hat\mu_{\bar g}}EE]\!]_F^E = 0$ for smooth $\bar g$ and $U\in H_0^2(\Omega,\mathbb{R}^{\tilde N\times\tilde N}_{\mathrm{sym}})$ there holds

$$\begin{aligned}&\left\|\sum_{F\supset E}\langle\sigma_E, [\![(SA_{g(t)})_{\hat\nu_{g(t)}\hat\mu_{g(t)}}EE]\!]_F^E\rangle\right\|_{L^1(E)}\\ &= \left\|\sum_{F\supset E}\langle\sigma_E, [\![(SA_{g(t)})_{\hat\nu_{g(t)}\hat\mu_{g(t)}}EE-(SA_{\bar g})_{\hat\nu_{\bar g}\hat\mu_{\bar g}}EE]\!]_F^E\rangle\right\|_{L^1(E)}\\ &\lesssim \|g_h-\bar g\|_{L^\infty(E)}\|\sigma_E\|_{L^2(E)}\|U\|_{L^2(E)}\end{aligned}$$

and further

$$\left|\int_E\sum_{F\supset E}\langle\sigma_E, [\![(SA_{g(t)})_{\hat\nu_{g(t)}\hat\mu_{g(t)}}EE]\!]_F^E\rangle\,\omega_E(g(t))\right| \lesssim \|g_h-g\|_{L^\infty(E)}\|U\|_{L^2(E)}\|\sigma_E\|_{L^2(E)}.$$

With the codimension 2 trace inequality

$$\|v\|^2_{L^2(E)} \leqslant C\left(h_T^{-2}\|v\|^2_{L^2(T)} + |v|^2_{H^1(T)} + h_T^2|v|^2_{H^2(T)}\right),\qquad E\subset T,$$

the claim follows. □

Proposition 5.17. *Let* $g(t)=\bar g+(g_h-\bar g)t$, $\sigma = g_h-\bar g$, *and* $U\in H_0^2(\Omega,\mathbb{R}^{\tilde N\times\tilde N}_{\mathrm{sym}})$. *There holds for all* $t\in[0,1]$ *for dimension* $N\geqslant 3$

$$\begin{aligned}\left|b(g(t);\sigma,U)\right| \lesssim &\left(1+\max_{T\in\mathscr{T}_h}h_T^{-2}\|g_h-\bar g\|_{L^\infty(T)}+\max_{T\in\mathscr{T}_h}h_T^{-1}\|g_h-\bar g\|_{W^{1,\infty}(T)}\right)\\ &\times |||g_h-\bar g|||_2\,\|U\|_{H^2}\end{aligned}$$

and for $N=2$

$$|b(g(t);\sigma,U)| \lesssim \left(1+\max_{T\in\mathscr{T}_h}h_T^{-1}\|g_h-\bar g\|_{L^\infty(T)}+\|g_h-\bar g\|_{W_h^{1,\infty}}\right)|||g_h-\bar g|||_2\,\|U\|_{H^2}.$$

Assume that $g_h=\mathfrak{I}_h^k\bar g$ *is an interpolant fulfilling assumptions* (5.1) *with a* $p\in[2,\infty]$. *Then for* $k\geqslant 0$ *for* $N=2$ *and* $k\geqslant 1$ *for* $N=3$

$$\left|b(g(t);\sigma,U)\right| \lesssim \left(\sum_{T\in\mathscr{T}_h}h_T^{p(k+1)}|\bar g|_{W^{k+1,p}(T)}\right)^{1/p}\|U\|_{H^2}.$$

Proof. Analogously to Proposition 5.12, combine Lemmas 5.14–5.16 and the approximation properties (5.1) of the interpolant together with (5.5). In the case of $N=2$ Lemma 5.16 does not need to be considered as the codimension 2 terms are zero. Together with the replacement of the $W^{1,\infty}$ by the L^∞ norm in Lemma 5.15 we obtain convergence also for piecewise constant metrics $k=0$. □

5.6. **Completing the convergence proofs.**

Proof of Theorem 5.2. For two dimensions $N = 2$ we use Proposition 5.17 noting that $a(\cdot;\cdot,\cdot) = 0$ in this case, see Lemma 6.1. In the case $N \geqslant 3$ combine Proposition 5.12 and Proposition 5.17. □

Proof of Corollary 5.3. This also follows from Proposition 5.12 and Proposition 5.17. □

6. The two and three dimensional cases

6.1. **Specialization to 2D.** In two dimensions, test space (3.1) consists of piecewise 0-forms, which are globally continuous, $\mathcal{U} = \wedge^0(\mathscr{T}) \cap C^0(\Omega)$, and the elements of (2.14) can be characterized, noting that $\star(X^\flat \wedge Y^\flat) = \omega(X,Y)$, via (3.2). Namely, they are all of the form in (2.16) for some $v \in \mathcal{U}$. In [23] the covariant incompatibility operator inc : $\wedge^1(T) \odot \wedge^1(T) \to \mathbb{R}$ has been defined in coordinates

$$\mathrm{inc}(\sigma) = \hat{\varepsilon}^{qi}\hat{\varepsilon}^{jk}\Big(\partial_{jq}\sigma_{ik} - \partial_q(\Gamma^m_{ji}\sigma_{mk}) - \Gamma^l_{lq}(\partial_j\sigma_{ik} - \Gamma^m_{ji}\sigma_{mk})\Big).$$

Additionally define the covariant curl of a 2-tensor in the right-handed frame $\{\hat{\tau},\hat{\nu}\}$ by $\mathrm{curl}(\sigma)(X) = (\nabla_{\hat{\tau}}\sigma)(\hat{\nu},X) - (\nabla_{\hat{\nu}}\sigma)(\hat{\tau},X)$. (A coordinate expression can be found in [23].)

Proposition 6.1. *The distributional densitized Riemann curvature tensor simplifies in 2D to the densitized distributional Gauss curvature (after rescaling by a factor* 4*)*

$$\widetilde{K\omega}(u) = \sum_{T\in\mathscr{T}}\int_T K u\,\omega_T + \sum_{F\in\mathring{\mathscr{F}}}\int_F [\![\kappa]\!]_F u\,\omega_F + \sum_{E\in\mathring{\mathscr{E}}}\Theta_E u(E), \qquad \textit{for all } u \in \mathring{\mathcal{U}}(\mathscr{T}),$$

and for the bilinear forms (3.37)*–*(3.38) *there holds for all* $\sigma \in \mathrm{Reg}(\mathscr{T})$ *and* $u \in \mathcal{U}$

$$\begin{aligned}
&a(g;\sigma,u) = 0,\\
&b(g;\sigma,u) = -2\sum_{T\in\mathscr{T}}\int_T \mathrm{inc}\,\sigma\,u\,\omega_T + 2\sum_{F\in\mathring{\mathscr{F}}}\int_F [\![\mathrm{curl}(\sigma)(\hat{\tau}) + \nabla_{\hat{\tau}}(\sigma_{\hat{\nu}\hat{\tau}})]\!]_F u\,\omega_F\\
&\qquad\qquad - 2\sum_{E\in\mathring{\mathscr{E}}}\sum_{F\supset E}[\![\sigma_{\hat{\nu}\hat{\mu}}]\!]^E_F u(E).
\end{aligned}$$

Especially, $b(g;\cdot,\cdot)$ *coincides with the distributional covariant incompatibility operator defined in* [23] *up to a factor* -2.

Proof. Let $A = -u\,\omega \otimes \omega$. The reduction to the stated expression of the distributional Gauss curvature was already shown in §2.6.

For proving that $a(g;\sigma,u) = 0$ we start with the first volume term of (3.37)

$$\begin{aligned}
\langle L^{(1)}_\sigma \mathcal{R}, A\rangle &= \mathcal{R}_{1212}{\sigma^1}_1 A^{1212} + \mathcal{R}_{1221}{\sigma^1}_1 A^{1221} + \mathcal{R}_{2112}{\sigma^2}_2 A^{2112} + \mathcal{R}_{2121}{\sigma^2}_2 A^{2121}\\
&= 2(\mathcal{R}_{1212}{\sigma^1}_1 A^{1212} + \mathcal{R}_{2112}{\sigma^2}_2 A^{2112})\\
&= 2\mathcal{R}_{1212}A^{1212}\,\mathrm{tr}(\sigma) = \frac{1}{2}\,\mathrm{tr}(\sigma)\,\langle \mathcal{R}, A\rangle,
\end{aligned}$$

which cancels with the second volume term of (3.37). Next, we consider the boundary terms recalling that $\mathbb{S}_F(\sigma) = \sigma_F - \mathrm{tr}(\sigma_F)g_F$

$$[\![\mathrm{I\!I}]\!] : \mathbb{S}_F(\sigma) : A_{F\hat{\nu}\hat{\nu}F} = (\sigma_{\hat{\tau}\hat{\tau}}[\![\kappa]\!] - [\![\kappa]\!]\sigma_{\hat{\tau}\hat{\tau}})A_{\hat{\tau}\hat{\nu}\hat{\nu}\hat{\tau}} = 0.$$

The claim therefore follows together with $\operatorname{tr}(\sigma_E) = 0$ on 0-dimensional vertices.

For proving the stated expression for $b(g;\sigma,u)$, we can verify as in [23] that

$$\langle \nabla^2\sigma, SA\rangle = -(\operatorname{inc}\sigma)\, u.$$

Further, on each facet F we have with $A_{\hat\tau\hat\nu\hat\nu\hat\tau} = u$ and $\kappa^{\hat\nu} = \mathrm{I\!I}^{\hat\nu}(\hat\tau,\hat\tau)$

$$\begin{aligned}
\langle [\![\sigma_{\hat\nu\hat\nu}\mathrm{I\!I} + (\nabla\sigma)_{F\hat\nu F} + \nabla(\hat\nu\lrcorner\sigma) - (\nabla\sigma)_{\hat\nu FF}]\!], A_{F\hat\nu\hat\nu F}\rangle&\\
&= [\![\sigma_{\hat\nu\hat\nu}\mathrm{I\!I}_{\hat\tau\hat\tau} + (\nabla_{\hat\tau}\sigma)_{\hat\nu\hat\tau} + \nabla_{\hat\tau}(\hat\nu\lrcorner\sigma)_{\hat\tau} - (\nabla_{\hat\nu}\sigma)_{\hat\tau\hat\tau}]\!]u\\
&= [\![\sigma_{\hat\nu\hat\nu}\,\kappa + (\nabla_{\hat\tau}\sigma)_{\hat\nu\hat\tau} - (\nabla_{\hat\nu}\sigma)_{\hat\tau\hat\tau} + \nabla_{\hat\tau}(\sigma_{\hat\nu\hat\tau}) - \sigma_{\hat\nu\hat\nu}\kappa]\!]u\\
&= [\![(\nabla_{\hat\tau}\sigma)_{\hat\nu\hat\tau} - (\nabla_{\hat\nu}\sigma)_{\hat\tau\hat\tau} + \nabla_{\hat\tau}(\sigma_{\hat\nu\hat\tau})]\!]u = [\![\operatorname{curl}(\sigma)(\hat\tau) + \nabla_{\hat\tau}(\sigma_{\hat\nu\hat\tau})]\!]u,
\end{aligned}$$

where we used that with the Leibnitz rule and $\nabla_{\hat\tau}\hat\tau = g(\nabla_{\hat\tau}\hat\tau,\hat\nu)\hat\nu = \kappa^{\hat\nu}\hat\nu$ there holds

$$\nabla_{\hat\tau}(\hat\nu\lrcorner\sigma)_{\hat\tau} = \nabla_{\hat\tau}(\sigma_{\hat\nu\hat\tau}) - \sigma(\hat\nu,\nabla_{\hat\tau}\hat\tau) = \nabla_{\hat\tau}(\sigma_{\hat\nu\hat\tau}) - \sigma_{\hat\nu\hat\nu}g(\nabla_{\hat\tau}\hat\tau,\hat\nu) = \nabla_{\hat\tau}(\sigma_{\hat\nu\hat\tau}) - \sigma_{\hat\nu\hat\nu}\kappa^{\hat\nu}.$$

The claim follows now with $A_{\hat\mu\hat\nu\hat\nu\hat\mu} = A_{\hat\tau\hat\nu\hat\nu\hat\tau} = u$. □

6.2. **Specialization to 3D.** On a 3-dimensional manifold, using a natural notion of the curl of a 2-tensor field, one can define a covariant incompatibility operator. In this subsection, we show that our N-dimensional incompatibility operator coincides with it when $N = 3$. We then show simplifications of our N-dimensional generalized curvature formula in 3D, and display coordinate formulas, which are also useful for 3D numerical experimentation (in Section 7).

We start by defining the covariant curl of 2-tensors, $\operatorname{curl} : \mathcal{T}^2_0(T) \to \mathcal{T}^2_0(T)$, by

$$(\operatorname{curl}\sigma)(X,Y) = \langle(\star d^\nabla L_\sigma)(X), Y\rangle, \qquad \sigma\in\mathcal{T}^2_0(T),\ X,Y\in\mathfrak{X}(T).$$

Here $L_\sigma \in \mathcal{T}^1_1(T)$ is the result of conversion of $\sigma\in\mathcal{T}^2_0(T)$ into an endomorphism per (3.12), and d^∇ is the *exterior covariant derivative*, recalled in (A.12), acting on L_σ considered as a vector-valued 1-form in $\wedge^1(T,\mathfrak{X}(T))$. Then $d^\nabla L_\sigma$ is in a vector-valued 2-form in $\wedge^2(T,\mathfrak{X}(T))$. Using the Hodge dual operator in 3D, we convert it to the vector-valued 1-form $\star d^\nabla L_\sigma$. (A similar 2D definition can be found in [23, Eq. (4.3) and Remark 4.1].) Combining two curl operations in succession, with an intervening transpose, we define the *three-dimensional covariant incompatibility* operator $\operatorname{inc} : \mathcal{S}(T) \to \mathcal{S}(T)$ by

$$(\operatorname{inc}\sigma)(X,Y) = (\operatorname{curl}(\operatorname{curl}\sigma)^T)(X,Y), \qquad \sigma\in\mathcal{S}(T),\ X,Y\in\mathfrak{X}(T),$$

where $(\operatorname{curl}\sigma)^T(X,Y) := (\operatorname{curl}\sigma)(Y,X)$. In coordinates, the covariant curl and incompatibility operator read

$$\begin{aligned}
[\operatorname{curl}\sigma]_{ij} &= \hat\varepsilon^{pql}g_{lj}(\partial_p\sigma_{iq} - \Gamma^m_{pi}\sigma_{mq}), \quad [\operatorname{curl}\sigma]_i^{\ j} = \hat\varepsilon^{pqj}(\partial_p\sigma_{iq} - \Gamma^m_{pi}\sigma_{mq}),\\
[\operatorname{inc}\sigma]_{ij} &= \hat\varepsilon^{pql}\hat\varepsilon^{rst}g_{lj}\Big((\partial_p g_{ti} - g_{ti}\Gamma^v_{vp})(\partial_r\sigma_{qs} - \Gamma^u_{rq}\sigma_{us})\\
&\qquad + g_{ti}\partial_p(\partial_r\sigma_{qs} - \Gamma^u_{rq}\sigma_{us}) - \Gamma^m_{pi}g_{tm}(\partial_r\sigma_{qs} - \Gamma^u_{rq}\sigma_{us})\Big),\\
[\operatorname{inc}\sigma]^{ij} &= \hat\varepsilon^{pqj}\big(\hat\varepsilon^{rsi}(\partial_p(\partial_r\sigma_{qs} - \Gamma^u_{rq}\sigma_{us}) - \Gamma^l_{lp}(\partial_r\sigma_{qs} - \Gamma^u_{rq}\sigma_{us}))\\
&\qquad + \hat\varepsilon^{rst}\Gamma^i_{pt}(\partial_r\sigma_{qs} - \Gamma^u_{rq}\sigma_{us})\big).
\end{aligned}$$

The Euclidean version of this 3D incompatibility operator (which can be obtained from the above by setting the metric g to the identity) has appeared extensively in the elasticity literature—see e.g., [1, 2, 11].

The next lemma shows the relationship between the above-defined 3D $\operatorname{inc}\sigma$ and the previously defined N-dimensional $\operatorname{Inc}\sigma$ of (4.4). Define the metric-dependent cross product of vector fields yielding 1-forms by

$$g(X \times Y, Z) = \omega(X, Y, Z), \qquad X, Y, Z \in \mathfrak{X}(\mathscr{T}). \tag{6.1a}$$

It reads in coordinates as $(X \times Y)^i = \hat{\varepsilon}^{ijk}(X^\flat)_j(Y^\flat)_k$. Similar formulas yield the tensor cross product [8] between two matrices $A, B \in \mathcal{T}^2_0(\mathscr{T})$ as well as between a matrix and a vector field $u \in \mathfrak{X}(\mathscr{T})$, as follows:

$$(A \times B)^{ij} := \hat{\varepsilon}^{ikl}\hat{\varepsilon}^{jmn}A_{km}B_{ln}, \qquad (A \times u)^{ij} := \hat{\varepsilon}^{jkl}A^i{}_k u_l. \tag{6.1b}$$

Lemma 6.2. *Let $T \in \mathscr{T}$, $U \in \mathcal{U}$, $A = \mathbb{A}U \in \mathcal{A}$, and $Q = I - \hat{\nu} \otimes \hat{\nu}$ the projection to the tangent space of a facet F of T with g-normal $\hat{\nu}$. Then, at a point p,*

$$\langle \operatorname{Inc}\sigma, A\rangle = \langle \operatorname{inc}\sigma, U\rangle, \qquad p \in T, \tag{6.2}$$

$$\langle (\nabla\sigma)_{F\hat{\nu}F} - (\nabla\sigma)_{\hat{\nu}FF}, A_{F\hat{\nu}\hat{\nu}F}\rangle = \langle Q(\operatorname{curl}\sigma)^T \times \hat{\nu}, U_F\rangle, \qquad p \in F. \tag{6.3}$$

Proof. To prove the stated identities it is helpful to work with Riemann normal coordinates (also called geodesic coordinates). At each point $p \in T$, a chart with these coordinates, which we denote by $\tilde{x}^i$, are such that [28, Proposition 5.24] the metric tensor with respect to $\tilde{x}^i$ becomes the identity and the Christoffel symbols vanish at the point p, i.e.,

$$\tilde{g}_{ij}|_p = \delta_{ij}, \qquad \tilde{\Gamma}_{ijk}|_p = 0.$$

We use a tilde to denote quantities in the Riemann normal coordinates.

To prove (6.2), we use (4.5) and Riemann normal coordinates. Starting with the left-hand side of the first identity and using Proposition 3.9, we get

$$\begin{aligned}\langle \operatorname{Inc}\sigma, A\rangle = -\langle \nabla^2\sigma, SA\rangle &= -\langle \nabla^2\sigma, S\mathbb{A}U\rangle \\ &= \big(\tilde{\partial}_l\tilde{\partial}_k\tilde{\sigma}_{ij} - \tilde{\partial}_l\tilde{\Gamma}_{ik\alpha}\tilde{\sigma}_{\alpha j} - \tilde{\partial}_l\tilde{\Gamma}_{kj\alpha}\tilde{\sigma}_{i\alpha}\big)\varepsilon^{lip}\varepsilon^{kjq}\tilde{U}_{pq} \\ &= \big(\tilde{\partial}_l\tilde{\partial}_k\tilde{\sigma}_{ij} - \tilde{\partial}_l\tilde{\Gamma}_{ik\alpha}\tilde{\sigma}_{\alpha j}\big)\varepsilon^{lip}\varepsilon^{kjq}\tilde{U}_{pq},\end{aligned}$$

which is, using the symmetry of U and the Christoffel symbols, the same as

$$\langle \operatorname{inc}\sigma, U\rangle = \big(\varepsilon^{ikl}\varepsilon^{jmn}\tilde{\partial}_k\tilde{\partial}_m\tilde{\sigma}_{ln} - \varepsilon^{jkl}\varepsilon^{i\alpha\beta}\tilde{\sigma}_{\gamma\beta}\tilde{\partial}_k\tilde{\Gamma}_{\alpha l\gamma}\big)\tilde{U}_{ij}.$$

To prove (6.3), we have for the left-hand side in Riemann normal coordinates and the coordinate system $\{\hat{\tau}_1, \hat{\tau}_2, \hat{\nu}\}$ with the g-orthonormal tangent vectors $\hat{\tau}_i$,

$$\begin{aligned}\langle (\nabla\sigma)_{F\hat{\nu}F} - (\nabla\sigma)_{\hat{\nu}FF}, A_{F\hat{\nu}\hat{\nu}F}\rangle &= \big((\tilde{\partial}_{\tilde{\tau}_i}\tilde{\sigma})_{\tilde{\tau}_j\tilde{\nu}} - (\tilde{\partial}_{\tilde{\nu}}\tilde{\sigma})_{\tilde{\tau}_i\tilde{\tau}_j}\big)\tilde{U}(\tilde{\nu} \times \tilde{\tau}_i, \tilde{\nu} \times \tilde{\tau}_j) \\ = \big((\partial_{\tilde{\tau}_1}\tilde{\sigma})_{\tilde{\nu}\tilde{\tau}_1} - (\partial_{\tilde{\nu}}\tilde{\sigma})_{\tilde{\tau}_1\tilde{\tau}_1}\big)&\tilde{U}_{\tilde{\tau}_2\tilde{\tau}_2} - \big((\partial_{\tilde{\tau}_1}\tilde{\sigma})_{\tilde{\nu}\tilde{\tau}_2} - (\partial_{\tilde{\nu}}\tilde{\sigma})_{\tilde{\tau}_1\tilde{\tau}_2}\big)\tilde{U}_{\tilde{\tau}_2\tilde{\tau}_1} \\ - \big((\partial_{\tilde{\tau}_2}\tilde{\sigma})_{\tilde{\nu}\tilde{\tau}_1} - (\partial_{\tilde{\nu}}\tilde{\sigma})_{\tilde{\tau}_2\tilde{\tau}_1}\big)&\tilde{U}_{\tilde{\tau}_1\tilde{\tau}_2} + \big((\partial_{\tilde{\tau}_2}\tilde{\sigma})_{\tilde{\nu}\tilde{\tau}_2} - (\partial_{\tilde{\nu}}\tilde{\sigma})_{\tilde{\tau}_2\tilde{\tau}_2}\big)\tilde{U}_{\tilde{\tau}_1\tilde{\tau}_1}\end{aligned}$$

and for the right-hand side written in the basis $\{\hat{\tau}_1, \hat{\tau}_2, \hat{\nu}\}$

$$\begin{aligned}&\langle Q\operatorname{curl}(\sigma)^T \times \hat{\nu}, U_F\rangle \\ &= \begin{pmatrix} (\tilde{\partial}_{\tilde{\tau}_2}\tilde{\sigma})_{\tilde{\tau}_2\tilde{\nu}} - (\tilde{\partial}_{\tilde{\nu}}\tilde{\sigma})_{\tilde{\tau}_2\tilde{\tau}_2} & (\tilde{\partial}_{\tilde{\nu}}\tilde{\sigma})_{\tilde{\tau}_1\tilde{\tau}_2} - (\tilde{\partial}_{\tilde{\tau}_2}\tilde{\sigma})_{\tilde{\tau}_1\tilde{\nu}} & 0 \\ (\tilde{\partial}_{\tilde{\nu}}\tilde{\sigma})_{\tilde{\tau}_2\tilde{\tau}_1} - (\tilde{\partial}_{\tilde{\tau}_1}\tilde{\sigma})_{\tilde{\tau}_2\tilde{\nu}} & (\tilde{\partial}_{\tilde{\tau}_1}\tilde{\sigma})_{\tilde{\tau}_1\tilde{\nu}} - (\tilde{\partial}_{\tilde{\nu}}\tilde{\sigma})_{\tilde{\tau}_1\tilde{\tau}_1} & 0 \\ 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \tilde{U}_{\tilde{\tau}_1\tilde{\tau}_1} & \tilde{U}_{\tilde{\tau}_1\tilde{\tau}_2} & 0 \\ \tilde{U}_{\tilde{\tau}_2\tilde{\tau}_1} & \tilde{U}_{\tilde{\tau}_2\tilde{\tau}_2} & 0 \\ 0 & 0 & 0 \end{pmatrix}.\end{aligned}$$

The claim follows by noting that the expressions coincide. □

Next, we proceed to examine the terms of the generalized Riemann curvature in 3D. We again use the mapping (3.2) to identify the test functions $A = \mathbb{A}U \in \mathcal{A}$ with $U \in \mathcal{U}$ via

$$\begin{aligned}(\mathbb{A}U)(X,Y,Z,W) &= \langle U, \star(X^\flat \wedge Y^\flat) \odot \star(W^\flat \wedge Z^\flat)\rangle \\ &= \langle U, \star(X^\flat \wedge Y^\flat) \otimes \star(W^\flat \wedge Z^\flat)\rangle \\ &= U\big((\star(X^\flat \wedge Y^\flat))^\sharp, (\star(W^\flat \wedge Z^\flat))^\sharp\big) \\ &= -U(X \times Y, Z \times W),\end{aligned} \tag{6.4}$$

where in the last step we have used the 3D cross product of (6.1a), which is easily seen to satisfy $X \times Y = (\star(X \wedge Y))^\sharp$.

Lemma 6.3. *When $N = 3$, the metric-independent test space equals the Regge space, i.e.,*

$$\mathring{\mathcal{U}} = \mathring{\mathrm{Reg}}(\mathscr{T}).$$

Proof. By the definition of $\mathcal{U}$ in (3.1), any $U \in \mathcal{U} = \wedge^1(\mathscr{T}) \odot \wedge^1(\mathscr{T})$ is a linear combination of symmetric dyadic products of coordinate 1-forms dx^i, namely

$$U = U_{ij}(dx^i \otimes dx^j + dx^j \otimes dx^i).$$

Any element of $\mathring{\mathrm{Reg}}(\mathscr{T})$ takes the same form. Since the interface continuity conditions and boundary conditions of $\mathring{\mathcal{U}}$ and $\mathring{\mathrm{Reg}}(\mathscr{T})$ match, they must be the same space. □

The curvature operator $\mathcal{Q}$ defined in Remark 3.8 simplifies to a symmetric 2-tensor in three dimensions. In coordinates, it reads, by (3.11),

$$\mathcal{Q}^{ij} = (\mathbb{A}^{-1}\mathcal{R})^{ij} = -\frac{1}{4}\hat{\varepsilon}^{i\,kl}\hat{\varepsilon}^{j\,mn}\mathcal{R}_{klmn}. \tag{6.5}$$

Motivated by (6.2), let us define a generalized 3D incompatibility operator $\widetilde{\mathrm{inc}}$ as a linear functional on the Regge space by

$$\widetilde{\mathrm{inc}\,\sigma}(U) = \widetilde{\mathrm{Inc}\,\sigma}(\mathbb{A}U), \qquad U \in \mathring{\mathrm{Reg}}(\mathscr{T}), \tag{6.6}$$

where $\widetilde{\mathrm{Inc}\,\sigma}$ is as in Definition 4.5.

Proposition 6.4. *The distributional densitized Riemann curvature tensor in 3D yields (after rescaling by a factor of 4) the following densitized distributional curvature $\widetilde{\mathcal{Q}\omega}$ as a functional on the Regge space:*

$$\widetilde{\mathcal{Q}\omega}(U) = \sum_{T\in\mathscr{T}}\int_T \langle \mathcal{Q}, U\rangle\,\omega_T - \sum_{F\in\mathring{\mathscr{F}}}\int_F \langle [\![\overline{\mathrm{I\!I}}]\!], U_F\rangle\,\omega_F + \sum_{E\in\mathring{\mathscr{E}}}\int_E \Theta_E\, U_{\hat{\tau}\hat{\tau}}\,\omega_E, \tag{6.7}$$

for all $U \in \mathring{\mathrm{Reg}}(\mathscr{T})$, where $\overline{\mathrm{I\!I}}^{\hat{\nu}} = \mathbb{S}_F\mathrm{I\!I}^{\hat{\nu}} = \mathrm{I\!I}^{\hat{\nu}} - H^{\hat{\nu}}g_F$ is the trace-reversed second fundamental form with $\mathbb{S}_F V = V_F - \mathrm{tr}(V_F)g_F$ the trace-reversed part of a 2-tensor V restricted to the facet F, and $\hat{\tau} = \hat{\nu} \times \hat{\mu}$ is a tangent vector along the edge E. The *bilinear forms* (3.37)–(3.38) *read, for all* $\sigma \in \mathrm{Reg}(\mathscr{T})$ *and* $U \in \mathring{\mathrm{Reg}}(\mathscr{T})$, *as follows:*

$$\begin{aligned}
a(g;\sigma,U) &= -2\sum_{T\in\mathscr{T}}\int_T \mathcal{Q} : \sigma : U\,\omega_T \\
&\quad - 2\sum_{F\in\mathring{\mathscr{F}}}\int_F [\![\mathrm{I\!I}]\!] : \mathbb{S}_F(\sigma) : \mathbb{S}_F(U)\,\omega_F \\
&\quad - 2\sum_{E\in\mathring{\mathscr{E}}}\int_E \Theta_E\,\sigma_{\hat\tau\hat\tau}\,U_{\hat\tau\hat\tau}\,\omega_E, \\
b(g;\sigma,U) &= -2\sum_{T\in\mathscr{T}}\int_T \langle \operatorname{inc}\sigma, U\rangle\,\omega_T \\
&\quad + 2\sum_{F\in\mathring{\mathscr{F}}}\int_F \langle [\![Q(\operatorname{curl}\sigma)^T \times \hat\nu - \sigma_{\hat\nu\hat\nu}\overline{\mathrm{I\!I}} - \mathbb{S}_F\nabla(\hat\nu\lrcorner\sigma)]\!], U_F\rangle\,\omega_F \\
&\quad - 2\sum_{E\in\mathring{\mathscr{E}}}\int_E \sum_{F\supset E} [\![\sigma_{\hat\nu\hat\mu}]\!]_F^E U_{\hat\tau\hat\tau}\,\omega_E.
\end{aligned}$$

Moreover, the latter expression is related to the generalized 3D covariant incompatibility operator of (6.6) *by*

$$b(g;\sigma,U) = -2\,\widetilde{\operatorname{inc}\sigma}(U). \tag{6.8}$$

Proof. To prove (6.7), we start with the identity of Theorem 3.4. Note that within each element, using (3.6) and the definition of $\mathcal{Q}$ in (3.8)

$$\langle \mathcal{R}, \mathbb{A}U\rangle = \langle \mathbb{A}\mathcal{Q}, \mathbb{A}U\rangle = \langle \star^{\odot 2}\mathcal{Q}, \star^{\odot 2}U\rangle = 4\langle \mathcal{Q}, U\rangle,$$

where the last identity followed from (A.11) with $N = 3$. This produces the first term on the right-hand side of (6.7). To obtain the facet term in (6.7), we use Proposition 3.9, (6.1b), and the identities $(\hat\nu\otimes\hat\nu)\times V = -\mathbb{S}_F V^T$ and $\langle V_F, \mathbb{S}_F W\rangle = \langle \mathbb{S}_F V, W_F\rangle$ for any 2-tensors V, W, which can be shown by a direct computation,

$$\begin{aligned}
\langle [\![\mathrm{I\!I}]\!], (\mathbb{A}U)_{\cdot\hat\nu\hat\nu\cdot}\rangle &= [\![\mathrm{I\!I}]\!]_{ij}[\mathbb{A}U]^{iklj}\hat\nu_k\hat\nu_l = -[\![\mathrm{I\!I}]\!]_{ij}\hat\varepsilon^{ik\,\alpha}\,\hat\varepsilon^{lj\,\beta}\,U_{\alpha\beta}\hat\nu_k\hat\nu_l \\
&= \langle [\![\mathrm{I\!I}]\!], (\hat\nu\otimes\hat\nu)\times U\rangle = -\langle [\![\mathrm{I\!I}]\!], \mathbb{S}_F U\rangle = \langle [\![\overline{\mathrm{I\!I}}]\!], U_F\rangle.
\end{aligned} \tag{6.9}$$

Finally, the codimension 2 term in (6.7) also follows by Proposition 3.9

$$(\mathbb{A}U)_{\hat\mu\hat\nu\hat\nu\hat\mu} = -\hat\varepsilon^{ij\,\alpha}\,\hat\varepsilon^{kl\,\beta}\,U_{\alpha\beta}\hat\mu_i\hat\nu_j\hat\nu_k\hat\mu_l = U_{\alpha\beta}\hat\tau^\alpha\hat\tau^\beta,$$

where $\hat\tau = \hat\nu\times\hat\mu$ is the tangent vector of the edge E.

Next, let us prove the stated expressions for a and b. By (6.4) and (6.5),

$$\begin{aligned}
\langle L_\sigma^{(1)}\mathcal{R}, \mathbb{A}U\rangle &= \hat\varepsilon_{ijo}\hat\varepsilon_{klp}\mathcal{Q}^{op}\sigma^i{}_a\hat\varepsilon^{ajm}\hat\varepsilon^{kln}U_{mn} \\
&= 2\big(\delta_i{}^a\delta_o{}^m - \delta_i{}^m\delta_o{}^a\big)\mathcal{Q}^{on}\sigma^i{}_aU_{mn} \\
&= 2\operatorname{tr}(\sigma)\langle \mathcal{Q}, U\rangle - 2\,\mathcal{Q} : \sigma : U,
\end{aligned}$$

where we used the identities $\varepsilon_{klp}\varepsilon^{kln} = 2\delta_p{}^n$ and $\varepsilon_{ijo}\varepsilon^{ajm} = \delta_i{}^a\delta_o{}^m - \delta_i{}^m\delta_o{}^a$. The first term cancels with

$$-\frac{1}{2}\operatorname{tr}(\sigma)\langle \mathcal{R}, \mathbb{A}U\rangle = -2\operatorname{tr}(\sigma)\langle \mathcal{Q}, U\rangle.$$

For the codimension 1 term we have similar to (6.9)

$$[\![\mathrm{I\!I}]\!] : \mathbb{S}_F(\sigma) : A_{F\hat\nu\hat\nu F} = -[\![\mathrm{I\!I}]\!] : \mathbb{S}_F(\sigma) : \mathbb{S}_F(U).$$

There holds $(\mathbb{A}U)_{\hat\mu\hat\nu\hat\nu\hat\mu} = U_{\hat\tau\hat\tau}$ and $\operatorname{tr}(\sigma_E) = \sigma_{\hat\tau\hat\tau}$, proving the expression of $a(g;\sigma,U)$. The stated expression for $b(g;\sigma,U)$ follows from Lemma 6.2 and

$$\langle \sigma_{\hat\nu\hat\nu}\mathrm{I\!I} + \nabla(\hat\nu \lrcorner \sigma), A_{F\hat\nu\hat\nu F}\rangle = -\langle [\![\sigma_{\hat\nu\hat\nu}\overline{\mathrm{I\!I}} + \mathbb{S}_F\nabla(\hat\nu\lrcorner\sigma)]\!], U_F\rangle.$$

Finally, (6.8) follows from (4.24). □

Remark 6.5 (Distributional incompatibility in the 3D Euclidean case). Assume that the triangulation $\mathscr{T}$ consists of non-curved simplices. In the Euclidean case, when the metric is the identity, Proposition 6.4 and (6.8), after simplifications, yield

$$\begin{aligned} \widetilde{\operatorname{inc}\sigma}(U) = &\sum_{T\in\mathscr{T}}\int_T \langle \operatorname{inc}\sigma, U\rangle\, dx \\ &- \sum_{F\in\mathring{\mathscr{F}}}\int_F \langle [\![Q(\operatorname{curl}\sigma)^T \times \nu - \mathbb{S}_F(\operatorname{grad}^F \sigma_\nu)]\!], U_F\rangle\, ds \\ &+ \sum_{E\in\mathring{\mathscr{E}}}\int_E \sum_{F\supset E} [\![\sigma_{\nu\mu}]\!]_F^E U_{\tau\tau}\, dl, \end{aligned} \tag{6.10}$$

where dx, ds, and dl are the volume, surface, and line elements, respectively, and all involved differential operators and tangent, normal, and conormal vectors are the standard Euclidean ones in 3D. Since inc is a constant-coefficient linear differential operator in the Euclidean case, it has a classical generalization as a distributional derivative when applied to a σ that is only piecewise smooth, given by

$$(\operatorname{inc}\sigma)^{\mathrm{dist}}(\varphi) = \int_\Omega \sigma : \operatorname{inc}\varphi\, dx$$

for all φ in $\mathcal{D}(\Omega)^{3\times3}$ with components in the Schwartz space of smooth compactly supported test functions. Observe that $(\operatorname{inc}\sigma)^{\mathrm{dist}}(\varphi)$ equals $\widetilde{\operatorname{inc}\sigma}(\varphi) = \widetilde{\operatorname{Inc}\sigma}(\mathbb{A}\varphi)$ by Theorem 4.1. Hence the linear functional in (6.10) gives the distributional inc when applied with $U = \varphi$. When σ is in the lowest order (piecewise constant) Regge space, equation (6.10) reduces to the formula for (Euclidean) distributional incompatibility of σ derived in [13].

7. Numerical examples

In this section we show that the theoretical convergence rates from Corollary 5.3 are sharp, in as far as can be confirmed by numerical experiments. All experiments were performed in the open source finite element software NGSolve[1] [36, 37], where the Regge elements are available.

We consider in dimension $N = 3$ the example proposed in [21] on the unit cube $\Omega = (-1,1)^3$ and the Riemannian metric tensor is induced by the embedding $(x,y,z) \mapsto (x,y,z,f(x,y,z))$, where $f(x,y,z) := \frac{1}{2}(x^2+y^2+z^2) - \frac{1}{12}(x^4+y^4+z^4)$. We will use the equivalent formulation of the curvature operator (6.7) and emphasize that the test function A and formulation (2.15) can also be used. The

[1] www.ngsolve.org

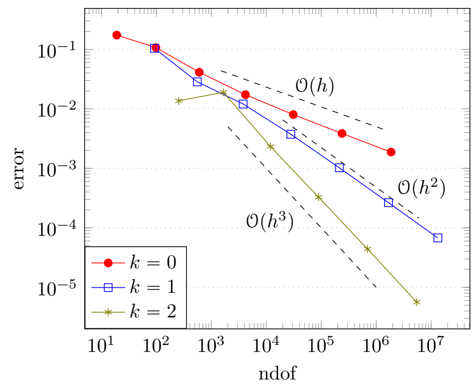

FIGURE 3. Convergence of the distributional curvature operator $\mathcal{Q}$ in the $H^{-2}(\Omega)$-norm for $N = 3$ with respect to the number of degrees of freedom (ndof) of $g_h \in \mathrm{Reg}_h^k$ for $k = 0, 1, 2$.

	$k = 0$		$k = 1$		$k = 2$	
h	Error	Order	Error	Order	Error	Order
$3.46 \cdot 10^{-0}$	$1.74 \cdot 10^{-1}$		$1.04 \cdot 10^{-1}$		$1.36 \cdot 10^{-2}$	
$1.75 \cdot 10^{-0}$	$1.07 \cdot 10^{-1}$	0.71	$2.84 \cdot 10^{-2}$	1.98	$1.89 \cdot 10^{-2}$	-0.48
$9.76 \cdot 10^{-1}$	$4.12 \cdot 10^{-2}$	1.64	$1.21 \cdot 10^{-2}$	1.38	$2.33 \cdot 10^{-3}$	3.72
$5.28 \cdot 10^{-1}$	$1.73 \cdot 10^{-2}$	1.41	$3.74 \cdot 10^{-3}$	1.82	$3.27 \cdot 10^{-4}$	2.94
$2.68 \cdot 10^{-1}$	$8.00 \cdot 10^{-3}$	1.14	$1.02 \cdot 10^{-3}$	2.04	$4.41 \cdot 10^{-5}$	2.99
$1.34 \cdot 10^{-1}$	$3.86 \cdot 10^{-3}$	1.06	$2.65 \cdot 10^{-4}$	2.00	$5.64 \cdot 10^{-6}$	3.11
$6.83 \cdot 10^{-2}$	$1.88 \cdot 10^{-3}$	1.07	$6.77 \cdot 10^{-5}$	2.01		

TABLE 1. Convergence of the distributional curvature operator $\mathcal{Q}$. Same as Figure 3, but in tabular form.

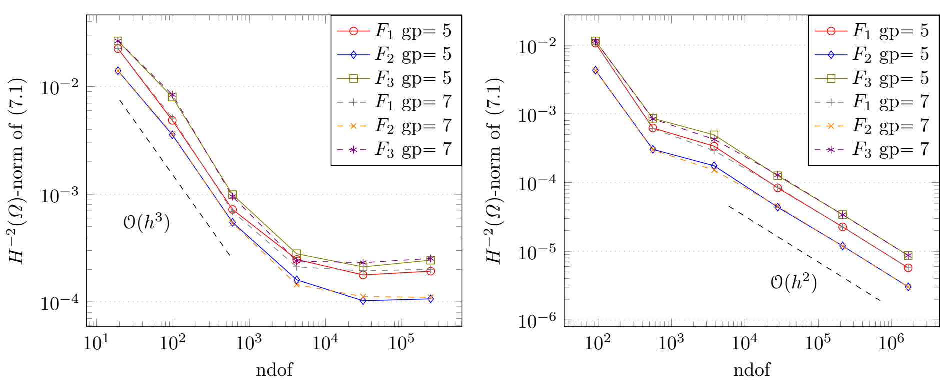

FIGURE 4. Convergence of the three functionals F_1, F_2, and F_3 in (7.1) in the $H^{-2}(\Omega)$-norm with respect to number of degrees of freedom (ndof) for 5 and 7 Gauss quadrature points (gp) in dimension $N = 3$. Left: $k = 0$. Right: $k = 1$.

components of the curvature operator read

$$\begin{aligned}
\mathcal{Q}_{xx} &= \frac{9(z^2-1)(y^2-1)}{\det(\bar{g})\big(q(x)+q(y)+q(z)+9\big)},\\
\mathcal{Q}_{yy} &= \frac{9(z^2-1)(x^2-1)}{\det(\bar{g})\big(q(x)+q(y)+q(z)+9\big)},\\
\mathcal{Q}_{zz} &= \frac{9(x^2-1)(y^2-1)}{\det(\bar{g})\big(q(x)+q(y)+q(z)+9\big)},\\
\mathcal{Q}_{xy} &= \mathcal{Q}_{xz} = \mathcal{Q}_{yz} = 0,
\end{aligned}$$

where $q(x) = x^2(x^2-3)^2$.

We compute the $H^{-2}(\Omega)$-norm of the error $f := \widetilde{\mathcal{Q}\omega}(g_h) - \mathcal{Q}\omega(\bar{g})$ by using that $\|f\|_{H^{-2}(\Omega,\mathbb{R}^{3\times 3}_{\mathrm{sym}})}$ is equivalent to $\|V\|_{H^{2}(\Omega,\mathbb{R}^{3\times 3}_{\mathrm{sym}})}$, where $V \in H^2_0(\Omega,\mathbb{R}^{3\times 3}_{\mathrm{sym}})$ solves the biharmonic equation $\Delta^2 V = f$ applied to each component. This equation will be solved numerically using the (Euclidean) Hellan–Herrmann–Johnson (HHJ) method [16] for each component of V (Although the HHJ method was originally defined only for two-dimensional domains, it is straight forward to extend it to arbitrary dimensions, see e.g. [29].). To avoid that the discretization error spoils the real error, we use for V_h two polynomial orders more than for $g_h \in \mathrm{Reg}^k_h$.

We will consider a structured mesh consisting of $6\cdot 2^{3i}$ tetrahedra, with $\tilde{h} = \max_T h_T = \sqrt{3}\,2^{1-i}$ (and minimal edge length 2^{1-i}) for $i = 0, 1, \dots$. We perturb each component of the inner mesh vertices by a random number drawn from a uniform distribution in the range $[-\tilde{h}\,2^{-3.5}, \tilde{h}\,2^{-3.5}]$ to avoid possible superconvergence due to mesh symmetries. As shown in Figure 3 and displayed in Table 1, we obtain linear convergence when $g_h \in \mathrm{Reg}^k_h$ has polynomial degree $k = 0$. For $k = 1$ and $k = 2$, higher convergence rates are obtained as expected. Therefore, Corollary 5.3 is sharp for $k \geqslant 1$. For $k = 0$ we observe numerically linear convergence, which is better than predicted by Corollary 5.3. Further investigations suggest, however, that the observed linear convergence for $k = 0$ is only pre-asymptotic. To test if Lemma 5.11, Lemma 5.16 (adapted to dimension $N = 3$), and the sum of both are sharp, we compute the $H^{-2}(\Omega)$-norm of the linear functionals

$$\begin{aligned}
&F_1 : U \mapsto \frac{1}{2}\int_0^1 \sum_{E\in\mathring{\mathscr{E}}} \int_E \sigma_{\hat{\tau}_{g(t)}\hat{\tau}_{g(t)}} \Theta_E(g(t)) U_{\hat{\tau}_{g(t)}\hat{\tau}_{g(t)}}\, \omega_E(g(t))\, dt,\\
&F_2 : U \mapsto -\frac{1}{2}\int_0^1 \sum_{E\in\mathring{\mathscr{E}}} \sum_{F\supset E} \int_E \sigma_{\hat{\tau}_{g(t)}\hat{\tau}_{g(t)}} [\![U_{\hat{\nu}_{g(t)}\hat{\mu}_{g(t)}}]\!]^E_F\, \omega_E(g(t))\, dt,\\
&F_3 = F_1 + F_2,
\end{aligned} \tag{7.1}$$

where we approximate the parameter integral by a Gauss quadrature with five and seven quadrature points. As depicted in Figure 4, the norm of this functional for the optimal-order interpolant g_h with $k = 0$ stagnates at about $2.5\cdot 10^{-4}$ after first converging with a cubic rate. The level of stagnation is lower than the overall error of about $2\cdot 10^{-3}$ for the finest grid, see Table 1. The loss of convergence is not due to approximation of the parameter integral, as the results change only marginally when going from five to seven integration points. Therefore, the lack of convergence stated by Corollary 5.3 is not yet visible in Figure 3. For linear elements

$k = 1$ quadratic convergence rates are obtained as expected from Lemma 5.11 and Lemma 5.16.

Appendix A. Summary of geometric notions used

Let (Ω, g) be an oriented Riemannian manifold with $\Omega \subset \mathbb{R}^N$. In this appendix (only), the metric g is smooth. Here we gather the standard notions in Riemannian geometry that we have used in previous sections, point out our choice of conventions or normalizations when multiple options exist, and provide references.

The value of a (k, l)-tensor field $\rho \in \mathcal{T}^k_l(\Omega)$ acting on k vectors $X_i \in \mathfrak{X}(\Omega)$ and l covectors $\mu_j \in \bigwedge^1(\Omega)$ is denoted by $\rho(X_1, \ldots, X_k, \mu_1, \ldots, \mu_l)$. Note that $\bigwedge^1(\Omega) = \mathcal{T}^1_0(\Omega)$ and $\mathfrak{X}(\Omega) = \mathcal{T}^0_1(\Omega)$. Note also that it is standard to extend the Levi–Civita connection ∇ from vector fields to tensor fields (see e.g., [28]) so that Leibniz rule holds, i.e. for $A \in \mathcal{T}^k_l(\Omega)$

$$
\begin{aligned}
(\nabla_X A)(Y_1, \ldots, Y_k, \alpha_1, \ldots, \alpha_l) &= X(A(Y_1, \ldots, Y_k, \alpha_1, \ldots, \alpha_l)) \\
&\quad - A(\nabla_X Y_1, \ldots, Y_k, \alpha_1, \ldots, \alpha_l) - \cdots - A(Y_1, \ldots, Y_k, \alpha_1, \ldots, \nabla_X \alpha_l).
\end{aligned}
$$

Notice that any symmetries of A are preserved by $\nabla_X A$. We define the $(k+1, l)$-tensor ∇A by $(\nabla A)(X, \ldots) = (\nabla_X A)(\ldots)$. Higher order operators can be defined inductively via $\nabla^{k+1} A = \nabla(\nabla^k A)$. We frequently use the second covariant derivative

$$
(\nabla^2_{X,Y} A)(\ldots) := (\nabla^2 A)(X, Y, \ldots) \tag{A.1}
$$

where we have selected the convention of placing the subscripts X, Y as the first two arguments (rather than the last two, as done in [28, p. 99]).

We use standard operations such as the tensor product $\otimes : \mathcal{T}^k_l(\Omega) \times \mathcal{T}^p_q(\Omega) \to \mathcal{T}^{k+p}_{l+q}(\Omega)$, the tangent to cotangent isomorphism $\flat : \mathfrak{X}(\Omega) \to \bigwedge^1(\Omega)$, the reverse operation $\sharp : \bigwedge^1(\Omega) \to \mathfrak{X}(\Omega)$, and the wedge product $\wedge$ with the normalization convention set to the so-called determinant convention, so that e.g.,

$$
\varphi \wedge \eta = \varphi \otimes \eta - \eta \otimes \varphi, \qquad \varphi, \eta \in \textstyle\bigwedge^1(\Omega). \tag{A.2}
$$

All these are exactly as in standard texts [28, 33, 41], where one can also find the definition of the Hodge dual (Hodge star) $\star : \bigwedge^k(\Omega) \to \bigwedge^{N-k}(\Omega)$, namely,

$$
g^{-1}(\star\eta, \mu)\, \omega = \eta \wedge \mu, \qquad \eta \in \textstyle\bigwedge^k(\Omega), \mu \in \bigwedge^{N-k}(\Omega). \tag{A.3}
$$

The unique volume-form over an oriented Riemannian manifold D is denoted by $\omega_D(g)$, and when $D = \Omega$, we simply abbreviate ω_Ω to ω, as in (A.3). There, as usual, the inner product $g^{-1}(\varphi^i, \phi^j) = g((\varphi^i)^\sharp, (\phi^j)^\sharp)$ between co-vectors φ^i and ϕ^j is extended to k-forms by $g^{-1}(\varphi^1 \wedge \cdots \wedge \varphi^k, \phi^1 \wedge \cdots \wedge \phi^k) = \det(g^{-1}(\varphi^i, \phi^j))$. One can prove, using the properties of the wedge product, that (A.3) implies

$$
g^{-1}(\star\eta, \mu) = (-1)^{k(N-k)} g^{-1}(\eta, \star\mu), \qquad \eta \in \textstyle\bigwedge^k(\Omega), \mu \in \bigwedge^{N-k}(\Omega). \tag{A.4}
$$

Given a g^{-1}-orthonormal co-vector basis e^i of matching orientation, (A.3) implies

$$
\star(e^1 \wedge \cdots \wedge e^k) = e^{k+1} \wedge \cdots \wedge e^N. \tag{A.5}
$$

Given a general $\eta \in \bigwedge^k(\Omega)$, a coordinate frame ∂_i and its coframe dx^j, the Hodge dual can be computed in terms of components $\eta_{i_1 \ldots i_k} = \eta(\partial_{i_1}, \ldots, \partial_{i_k})$ to get

$$
\star\eta = \frac{\sqrt{\det g}}{(N-k)!k!}\, \eta_{m_1 \ldots m_k} g^{m_1 i_1} \ldots g^{m_k i_k} \varepsilon_{i_1 \ldots i_k j_1 \ldots j_{N-k}}\, dx^{j_1} \wedge \cdots \wedge dx^{j_{N-k}}. \tag{A.6}
$$

In particular,

$$\star(dx^p \wedge dx^q) = \frac{\sqrt{\det g}}{(N-2)!}\, g^{rp} g^{sq}\, \varepsilon_{rsj_1\dots j_{N-2}} dx^{j_1} \wedge \cdots \wedge dx^{j_{N-2}}. \tag{A.7}$$

Using (A.6), one can prove that for vector fields $X_i, Y_j \in \mathfrak{X}(\Omega)$,

$$\star(X_1^\flat \wedge X_2^\flat \wedge \cdots \wedge X_k)(Y_1, \dots, Y_{N-k}) = \omega(X_1, X_2, \dots, X_k, Y_1, \dots, Y_{N-k}). \tag{A.8}$$

The notation $\langle X, Y\rangle := g(X,Y)$ denotes the g-inner product for two vector fields $X, Y \in \mathfrak{X}(\Omega)$, as well as its extension to general tensors $A, B \in \mathcal{T}^k_l(\Omega)$ through tensor product compatibility. Then, using coordinates of k-covariant tensors A and B,

$$\langle A, B\rangle = A_{i_1\dots i_k} g^{i_1 j_1} \dots g^{i_k j_k} B_{j_1\dots j_k}. \tag{A.9}$$

Note that for k-forms $\eta, \varphi \in \bigwedge^k(\Omega)$, this inner product and previously mentioned g^{-1} inner product are related by [28, Exercise 2-17]

$$g^{-1}(\eta, \varphi) = \frac{1}{k!}\langle \eta, \varphi\rangle. \tag{A.10}$$

It is easy to see from (A.3) that the Hodge dual is an isometry in the g^{-1} inner product. Hence (A.10) implies that for k-forms η, φ, we have $\langle \star\eta, \star\varphi\rangle = (N-k)!\, g^{-1}(\star\eta, \star\varphi) = (N-k)!\, g^{-1}(\eta, \varphi)$. Therefore

$$\langle \star\eta, \star\varphi\rangle = \frac{(N-k)!}{k!}\langle \eta, \varphi\rangle, \qquad \eta, \varphi \in \bigwedge\nolimits^k(\Omega), \tag{A.11}$$

showing that the Hodge dual is a quasi-isometry in the tensor inner product.

The exterior covariant derivative, see e.g. [17, p. 35],

$$d^\nabla : \bigwedge\nolimits^k(\Omega, \mathfrak{X}(\Omega)) \to \bigwedge\nolimits^{k+1}(\Omega, \mathfrak{X}(\Omega)) \tag{A.12}$$

extends the exterior derivative d from differential forms to e.g. vector-valued differential forms. For $k = 0$ it coincides with the exterior derivative, $d^\nabla = d$, and the exterior covariant derivative fulfills the Leibnitz rule

$$d^\nabla(\alpha \wedge \eta) = d\alpha \wedge \eta + (-1)^k \alpha \wedge d^\nabla \eta, \qquad \text{for all } \alpha \in \bigwedge\nolimits^k(\Omega),\ \eta \in \bigwedge\nolimits^l(\Omega, \mathfrak{X}(\Omega)),$$

allowing also for an inductive definition of d^∇. The Hodge dual in (A.3) can be readily extended to $\star : \wedge^k(\Omega, \mathfrak{X}(\Omega)) \to \wedge^{N-k}(\Omega, \mathfrak{X}(\Omega))$ by noting that

$$\wedge^k(\Omega, \mathfrak{X}(\Omega)) \simeq \wedge^k(\Omega) \otimes \mathfrak{X}(\Omega) \text{ and } \wedge^{N-k}(\Omega, \mathfrak{X}(\Omega)) \simeq \wedge^{N-k}(\Omega) \otimes \mathfrak{X}(\Omega),$$

and performing the standard Hodge star operation on the alternating part.

The covariant divergence of a tensor $A \in \mathcal{T}^k_0(\Omega)$, with $k \geqslant 1$, is defined as the trace of the covariant derivative of A in its first two components, $\mathrm{div} A := \mathrm{tr}_{12} \nabla A$. (Note that with this convention, in components, the divergence is applied to the first index of A.) We neglect the subscripts of the trace operator when there is no possibility of confusion. Equivalently, given a g-orthonormal basis e_i,

$$(\mathrm{div} A)(X_1, X_2, \dots) = \sum_{i=1}^N (\nabla A)(e_i, e_i, X_1, X_2, \dots). \tag{A.13}$$

for any $X_i \in \mathfrak{X}(\Omega)$.

Next, recall the classical Stokes theorem [27, Theorem 16.11] for $(N-1)$-forms. Applying it to the Hodge dual of a 1-form α, we get the divergence theorem for smooth 1-forms on Riemannian manifolds:

$$\int_{\Omega}(\operatorname{div}\alpha)\,\omega_{\Omega} = -\int_{\partial\Omega}\alpha(\hat{\nu})\,\omega_{\partial\Omega}, \qquad \text{for all } \alpha \in {\textstyle\bigwedge}^1(\Omega), \tag{A.14}$$

where $\hat{\nu}$ is the inward g-normalized unit normal of the boundary $\partial\Omega$ with the induced orientation. (Further standard assumptions needed for the existence of the integrals in the Stokes theorem, such as either the support of α or Ω is compact, are tacitly placed throughout.) Now, for arbitrary tensors $A \in \mathcal{T}_0^{k+1}(\Omega)$ and $B \in \mathcal{T}_0^k(\Omega)$, we can produce a 1-form by $\theta(X) = \langle X \lrcorner A, B\rangle$, using the standard interior product

$$(X \lrcorner A)(Y,\dots) = A(X, Y, \dots). \tag{A.15}$$

Then applying (A.14) to θ, we obtain the integration by parts formula

$$\int_{\Omega}\langle A, \nabla B\rangle \omega_{\Omega} = -\int_{\Omega}\langle \operatorname{div}A, B\rangle \omega_{\Omega} - \int_{\partial\Omega}\langle A, \hat{\nu}^\flat \otimes B\rangle \omega_{\partial\Omega}. \tag{A.16}$$

Next, consider an $(N-1)$-dimensional submanifold F of Ω with unit normal vector $\hat{\nu}$ and let $\{\hat{\tau}_1, \dots, \hat{\tau}_{N-1}, \hat{\nu}\}$ be an oriented g-orthonormal frame on F. Then, the surface divergence on F of any $A \in \mathcal{T}_0^k(\Omega)$, $k \geqslant 1$, can be calculated using this basis after omitting the last summand in (A.13), i.e.,

$$(\operatorname{div}^F A)(X_1, X_2, \dots) = \sum_{i=1}^{N-1}(\nabla A)(\tau_i, \tau_i, X_1, X_2, \dots) \tag{A.17}$$

for any $X_i \in \mathfrak{X}(\Omega)$. It equals the trace of

$$\nabla^F A := \nabla A - \hat{\nu}^\flat \otimes \nabla_{\hat{\nu}} A. \tag{A.18}$$

This can also be expressed using the Q in (2.2): extending $QX = (\mathrm{id} - \hat{\nu} \otimes \hat{\nu}^\flat)X$ on vectors X to 1-forms η by $Q\eta = (\mathrm{id} - \hat{\nu}^\flat \otimes \hat{\nu})\eta$, we see that for the covariant k-tensor A, $\nabla^F A$ is obtained by projecting the first argument, or in coordinates, $(\nabla^F A)_{ji_1\dots i_k} = Q_j{}^k \nabla_k A_{i_1\dots i_k}$.

Integration by parts formula on F, in contrast to (A.16), additionally involves a term with the mean curvature $H^{\hat{\nu}}$ of F. To see this, expressing the second fundamental form (in (2.7)) as $\mathrm{I\!I}^{\hat{\nu}} = -\nabla^F \hat{\nu}^\flat$, the mean curvature of F is given by

$$H^{\hat{\nu}} = \operatorname{tr}(\mathrm{I\!I}^{\hat{\nu}}) = -\sum_{i=1}^{N-1} g(\nabla_{\hat{\tau}_i}\hat{\nu}, \hat{\tau}_i) = -\operatorname{div}^F \hat{\nu}^\flat. \tag{A.19}$$

Here, we have used the sign convention that makes $H^{\hat{\nu}}$ positive for a sphere with an inward pointing normal vector. To obtain a surface integration by parts formula for 1-forms α on F, we split $\alpha = \alpha_F + \alpha(\hat{\nu})\hat{\nu}^\flat$ where $\alpha_F = \sum_{i=1}^{N-1}\alpha(\hat{\tau}_i)\hat{\tau}_i^\flat$ represents the form restricted to the surface. By (A.14) applied to α_F, we have

$$\int_F (\operatorname{div}^F \alpha_F)\,\omega_F = \int_{\partial F}\alpha(\hat{\mu})\,\omega_{\partial F}, \tag{A.20}$$

where $\hat{\mu}$ denotes the *inward*-pointing g-normalized conormal vector on ∂F. Since the splitting of α implies

$$\operatorname{div}^F \alpha = \operatorname{div}^F \alpha_F + \alpha(\hat{\nu})\operatorname{div}^F \hat{\nu}^\flat,$$

equations (A.19) and (A.20) yield

$$\int_F (\mathrm{div}^F \alpha)\, \omega_F = -\int_F H^{\hat\nu}\, \alpha(\hat\nu)\, \omega_F - \int_{\partial F} \alpha(\hat\mu)\, \omega_{\partial F}, \qquad \text{for all } \alpha \in \bigwedge^1(\Omega). \tag{A.21}$$

Now, by an argument similar to what we used to go from (A.14) to (A.16), we obtain the following surface integration by parts formula from (A.21):

$$\begin{aligned} \int_F \langle A, \nabla^F B\rangle \omega_F = &-\int_F \langle \mathrm{div}^F A, B\rangle \omega_F - \int_{\partial F} \langle A, \hat\mu^\flat \otimes B\rangle \omega_{\partial F} \\ &- \int_F H^{\hat\nu}\, \langle A, \hat\nu^\flat \otimes B\rangle \omega_F, \end{aligned} \tag{A.22}$$

for any smooth tensors $A \in \mathcal{T}_0^{k+1}(\Omega)$ and $B \in \mathcal{T}_0^k(\Omega)$.

Acknowledgments

This work was supported in part by the Austrian Science Fund (FWF) project 10.55776/F65 and the National Science Foundation (USA) grant DMS-2409900.

References

[1] Amstutz, S., and Van Goethem, N. Existence and asymptotic results for an intrinsic model of small-strain incompatible elasticity. *Discrete & Continuous Dynamical Systems - B 25*, 10 (2020), 3769–3805.

[2] Arnold, D. N., Awanou, G., and Winther, R. Finite elements for symmetric tensors in three dimensions. *Math. Comp. 77*, 263 (2008), 1229–1251.

[3] Arnold, D. N., Falk, R., and Winther, R. Finite element exterior calculus: from Hodge theory to numerical stability. *Bulletin of the American Mathematical Society 47*, 2 (2010), 281–354.

[4] Arnold, D. N., Falk, R. S., and Winther, R. Finite element exterior calculus, homological techniques, and applications. *Acta numerica 15* (2006), 1–155.

[5] Arnold, D. N., and Hu, K. Complexes from complexes. *Foundations of Computational Mathematics 21*, 6 (2021), 1739–1774.

[6] Barrett, J. W., Oriti, D., and Williams, R. M. Tullio Regge's legacy: Regge calculus and discrete gravity. *arXiv preprint arXiv:1812.06193* (2018).

[7] Berchenko-Kogan, Y., and Gawlik, E. S. Finite element approximation of the Levi-Civita connection and its curvature in two dimensions. *Foundations of Computational Mathematics* (2022).

[8] Bonet, J., Gil, A. J., and Ortigosa, R. On a tensor cross product based formulation of large strain solid mechanics. *International Journal of Solids and Structures 84* (2016), 49–63.

[9] Borrelli, V., Cazals, F., and Morvan, J.-M. On the angular defect of triangulations and the pointwise approximation of curvatures. *Computer Aided Geometric Design 20*, 6 (2003), 319–341.

[10] Cheeger, J., Müller, W., and Schrader, R. On the curvature of piecewise flat spaces. *Communications in Mathematical Physics 92*, 3 (1984), 405–454.

[11] Christiansen, S., Gopalakrishnan, J., Guzmán, J., and Hu, K. A discrete elasticity complex on three-dimensional Alfeld splits. *Numer. Math. 156* (2024), 159–204.

[12] Christiansen, S. H. A characterization of second-order differential operators on finite element spaces. *Mathematical Models and Methods in Applied Sciences 14*, 12 (2004), 1881–1892.

[13] Christiansen, S. H. On the linearization of Regge calculus. *Numerische Mathematik 119*, 4 (2011), 613–640.

[14] Christiansen, S. H. Exact formulas for the approximation of connections and curvature. *arXiv preprint arXiv:1307.3376* (2013).

[15] Christiansen, S. H. On the definition of curvature in Regge calculus. *IMA Journal of Numerical Analysis* (2024), drad095.

[16] Comodi, M. I. The Hellan–Herrmann–Johnson method: Some new error estimates and post-processing. *Mathematics of Computation 52*, 185 (1989), 17–29.

[17] Donaldson, S. K., and Kronheimer, P. B. *The Geometry of Four-Manifolds*. Oxford university press, New York, 1990.

[18] Fritz, H. Isoparametric finite element approximation of Ricci curvature. *IMA Journal of Numerical Analysis 33*, 4 (2013), 1265–1290.

[19] Fritz, H. Numerical Ricci–DeTurck flow. *Numerische Mathematik 131*, 2 (2015), 241–271.

[20] Gawlik, E. S. High-order approximation of Gaussian curvature with Regge finite elements. *SIAM Journal on Numerical Analysis 58*, 3 (2020), 1801–1821.

[21] Gawlik, E. S., and Neunteufel, M. Finite element approximation of scalar curvature in arbitrary dimension. *arXiv preprint arXiv:2301.02159* (2023).

[22] Gawlik, E. S., and Neunteufel, M. Finite element approximation of the Einstein tensor. *arXiv preprint arXiv:2310.18802* (2023).

[23] Gopalakrishnan, J., Neunteufel, M., Schöberl, J., and Wardetzky, M. Analysis of curvature approximations via covariant curl and incompatibility for Regge metrics. *The SMAI Journal of computational mathematics 9* (2023), 151–195.

[24] Hamilton, R. S. Four-manifolds with positive curvature operator. *Journal of Differential Geometry 24*, 2 (1986), 153–179.

[25] Hauret, P., and Hecht, F. A discrete differential sequence for elasticity based upon continuous displacements. *SIAM Journal on Scientific Computing 35*, 1 (2013), B291–B314.

[26] Israel, W. Singular hypersurfaces and thin shells in general relativity. *Il Nuovo Cimento B (1965-1970) 44*, 1 (1966), 1–14.

[27] Lee, J. M. *Introduction to Smooth Manifolds*, 2 ed. Springer, 2012.

[28] Lee, J. M. *Introduction to Riemannian Manifolds*, second ed. Springer, 2018.

[29] Li, L. *Regge Finite Elements with Applications in Solid Mechanics and Relativity*. PhD thesis, University of Minnesota, 2018.

[30] Liu, D., and Xu, G. Angle deficit approximation of Gaussian curvature and its convergence over quadrilateral meshes. *Computer-Aided Design 39*, 6 (2007), 506–517.

[31] Neunteufel, M. *Mixed Finite Element Methods for Nonlinear Continuum Mechanics and Shells*. PhD thesis, TU Wien, 2021.

[32] Neunteufel, M., and Schöberl, J. Avoiding membrane locking with Regge interpolation. *Computer Methods in Applied Mechanics and Engineering 373* (2021), 113524.

[33] Petersen, P. *Riemannian Geometry*, third ed. Springer, 2016.

[34] Regge, T. General relativity without coordinates. *Il Nuovo Cimento (1955-1965) 19*, 3 (1961), 558–571.

[35] Regge, T., and Williams, R. M. Discrete structures in gravity. *Journal of Mathematical Physics 41*, 6 (2000), 3964–3984.

[36] Schöberl, J. NETGEN an advancing front 2D/3D-mesh generator based on abstract rules. *Computing and Visualization in Science 1*, 1 (1997), 41–52.

[37] Schöberl, J. C++ 11 implementation of finite elements in NGSolve. *Institute for Analysis and Scientific Computing, Vienna University of Technology* (2014).

[38] Sorkin, R. Time-evolution problem in Regge calculus. *Phys. Rev. D 12* (1975), 385–396.

[39] Strichartz, R. S. Defining Curvature as a Measure via Gauss–Bonnet on Certain Singular Surfaces. *The Journal of Geometric Analysis 30*, 1 (2020), 153–160.

[40] Topping, P. *Lectures on the Ricci Flow*, vol. 325. Cambridge University Press, 2006.

[41] Tu, L. W. *Differential Geometry: Connections, Curvature, Characteristic Classes*. Springer, 2017.

[42] Williams, R. M., and Tuckey, P. A. Regge calculus: a brief review and bibliography. *Classical and Quantum Gravity 9*, 5 (1992), 1409–1422.

[43] Xu, G. Convergence analysis of a discretization scheme for Gaussian curvature over triangular surfaces. *Computer Aided Geometric Design 23*, 2 (2006), 193–207.

[44] Xu, Z., and Xu, G. Discrete schemes for Gaussian curvature and their convergence. *Computers & Mathematics with Applications 57*, 7 (2009), 1187–1195.

Portland State University, PO Box 751, Portland OR 97201, USA
Email address: gjay@pdx.edu

Portland State University, PO Box 751, Portland OR 97201, USA
Email address: mneunteu@pdx.edu

Institute for Analysis and Scientific Computing, TU Wien, Wiedner Hauptstr. 8-10, 1040 Wien, Austria
Email address: joachim.schoeberl@tuwien.ac.at

Institute of Numerical and Applied Mathematics, University of Göttingen, Lotzestr. 16-18, 37083 Göttingen, Germany
Email address: email: wardetzky@math.uni-goettingen.de