

THE JOHNSON-MERCIER ELASTICITY ELEMENT IN ANY DIMENSIONS

JAY GOPALAKRISHNAN, JOHNNY GUZMÁN, AND JEONGHUN J. LEE

ABSTRACT. Mixed methods for linear elasticity with strongly symmetric stresses of lowest order are studied in this paper. On each simplex, the stress space has piecewise linear components with respect to its Alfeld split (which connects the vertices to barycenter), generalizing the Johnson–Mercier two-dimensional element to higher dimensions. Further reductions in the stress space in the three-dimensional case (to 24 degrees of freedom per tetrahedron) are possible when the displacement space is reduced to local rigid displacements. Proofs of optimal error estimates of numerical solutions and improved error estimates via postprocessing and the duality argument are presented.

1. INTRODUCTION

The classical Hellinger–Reissner variational principle of linear elasticity yields saddle point problems simultaneously seeking the stress tensor field and the displacement vector field. A standard finite element discretization of the Hellinger–Reissner formulation then needs two finite element spaces, namely a stress space Σ_h of symmetric matrix-valued piecewise polynomials whose row-wise divergence is square integrable, and a displacement space V_h of vector fields with square-integrable components. Decades of research have illuminated the nontrivial difficulties in finding such a pair of spaces, $\Sigma_h \times V_h$, which also satisfy the Babuška–Brezzi stability conditions of saddle point problems.

To the best of our knowledge, the first such finite element space $\Sigma_h \times V_h$ with rigorous mathematical analysis was presented in [32] for the two-dimensional elastostatic problem (and the element itself appeared earlier in [40]). This finite element on triangular meshes, often called the Johnson–Mercier element, uses a Clough–Tocher refinement [19, 18] (also known as HCT refinement [17]) of a given mesh to define the shape functions of the discrete stress space Σ_h , whereas the discrete displacement space V_h is composed of piecewise linear functions defined on the original mesh. More precisely, Σ_h is the space of symmetric tensors such that divergence of each of its rows is square-integrable, and each of its components is piecewise linear on the Clough–Tocher refinement. A family of higher order elasticity elements in two dimensions with the Clough–Tocher refinement of triangular meshes was developed in [6], where the

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displacement spaces V_h also have piecewise polynomials on the Clough–Tocher refinement.

The goal of this work is an investigation of higher dimensional versions of these works on Alfeld splits, the natural generalization of Clough–Tocher splits to tetrahedra. At the outset, we were unaware that an extension of the Johnson–Mercier element to three dimensions (3D) was already done in 1982 by Křížek in [33]. This is even before Alfeld’s work [1] after which the tetrahedral split was known as the Alfeld split. A cursory literature search gives the impression that Křížek’s work is largely unknown at present: it is not mentioned in influential previous studies like [2, 29, 30, 26, 34, 31]. Nonetheless, a proof of unisolvency of the 3D version of the Johnson–Mercier stress element and error estimates for stress are presented in [33]. In this paper, we present a different proof of unisolvency of the same 3D stress element. We show that our new technique of proof generalizes to give a stress element in higher dimensions. Even in 3D, the new understanding helps us develop a reduced stress element with only 24 degrees of freedom per element, which to our knowledge, is the simplest conforming symmetric stress element known currently. We also prove error estimates for the displacement, prove superconvergence estimates by duality argument, show robustness in the incompressible limit, and develop a postprocessing to enhance convergence rates. We also study the method obtained by pairing the same stress space with piecewise constant displacements with respect to the Alfeld refinement (instead of the piecewise linear displacement spaces that Johnson and Mercier used).

Our approach is based on the analysis of [32, 6], rather than that of [33]. The unisolvency proofs in [32, 6] of stress finite elements Σ_h proceed by characterizing the divergence-free subspace of Σ_h that have vanishing normal components on the edges of a triangular element. They show that this subspace must consist of the Airy stress tensor of C^1 splines on the Clough–Tocher split. In the more modern language of finite element theory, this fits in an exact local discrete complex of the corresponding spaces with homogeneous boundary conditions. Note that $\text{div } \Sigma_h$ is not a subspace of the displacement space V_h used in the Johnson–Mercier element, so one would have to replace V_h with piecewise constant displacements on the Clough–Tocher triangulation to relate it to the complex. In contrast, the 3D analysis in [33] utilizes special geometric properties of tetrahedra and inter-element continuity of polynomials for unisolvency proof, an argument that seems difficult to generalize to higher dimensions. For our extension of Johnson–Mercier element in any dimensions, the key, borrowed from developments in finite element exterior calculus, is to exploit an identity used in the BGG resolution [3, 13, 21]. We give an elementary proof of this identity in arbitrary dimensions. Although we have chosen to present this generalization without using the language of exterior calculus, it is indeed motivated by the considerations to make the local stress space fit into an exact discrete elasticity complex. Indeed, similar ideas were critical in [14] where the first 3D discrete elasticity complex was developed on Alfeld splits. Since we only need to focus on the last part of such an N -dimensional complex to obtain the generalization of the Johnson–Mercier element, a complete study of the full N -dimensional complex is outside the scope of this paper.

Composite conforming stress elements are interesting because they promise liberation from the necessity of vertex degrees of freedom when using polynomial spaces on unsplit elements [2]. Due to complications with supersmoothness, this promise was not realized on Alfeld splits in [14], but was realized on the more involved Worsley-Farin splits (which splits a tetrahedron to 12 subtetrahedra) later in [26]. Through our improved understanding of the lowest order case, in this paper we are finally able to provide a 3D stress element on Alfeld splits without vertex and edge degrees of freedom. Even in higher dimensions, our composite stress element does not have degrees of freedom on subsimplices of codimension greater than one. In particular, this allows for a natural facet-based hybridization. Another nice property of our elasticity element is that only low-degree piecewise polynomials are used for the stress finite element. Most symmetric conforming stress elements for 3D elasticity contain piecewise quadratic or higher degree polynomials for stress (although [26] is an exception). Only some reduced elements have the stress shape functions which do not contain all quadratic polynomials [9], but contains other high polynomial degree shape functions. In this paper, even in N -dimensions, only piecewise linear polynomials are used for the stress.

The paper is organized as follows. We begin by defining the elasticity problem in Section 2. There we present our unisolvency proof of the 3D Johnson–Mercier element. We then analyze the stability and the error estimates of numerical solutions with the Johnson–Mercier element. Optimal error estimates of numerical solutions and superconvergent error estimates of post-processed solutions are proved. In Section 3, another numerical method with the 3D Johnson–Mercier stress element paired with piecewise constant displacement, is proposed and analyzed. We then present a reduced finite element pair in Section 4 and prove its error estimates. The connection of the Johnson–Mercier element and linear elasticity finite element methods with weakly symmetric stress is discussed in Section 5. Generalization of the Johnson–Mercier element for any N -dimensions ($N \geq 2$) is given in Section 6. Finally, a further reduced finite element pair and its error estimates are discussed in Appendix A.

2. THE JOHNSON-MERCIER ELEMENT FOR 3D ELASTICITY

2.1. Notation. For a measurable set $D \subset \mathbb{R}^N$ ($N \geq 2$), let $L^2(D)$ denote the Lebesgue space with N -dimensional Lebesgue measure. For a domain $D \subset \mathbb{R}^N$ with Lipschitz boundary, $H^k(D)$, $k \geq 1$, denotes the standard Sobolev space [22] of functions in $L^2(D)$ all of whose derivatives of up to k th order are also in $L^2(D)$. Let \mathbb{M} denote the space of $N \times N$ matrices. For $M \in \mathbb{M}$, let $\text{sym } M = \frac{1}{2}(M + M')$, $\text{skw } M = \frac{1}{2}(M - M')$, and let $\mathbb{S} = \text{sym } \mathbb{M}$ and $\mathbb{K} = \text{skw}(\mathbb{M})$. For vector spaces \mathbb{X} such as $\mathbb{R}, \mathbb{R}^N, \mathbb{M}, \mathbb{S}, \mathbb{K}$, etc., we let $L^2(\Omega, \mathbb{X})$ denote the space of \mathbb{X} -valued functions whose components are in $L^2(\Omega)$. The inner product of $L^2(D, \mathbb{X})$ and the corresponding norm are denoted $(\cdot, \cdot)_D$ and $\|\cdot\|_{L^2(D, \mathbb{X})}$, respectively. Similarly, the \mathbb{X} -valued Sobolev spaces and norms are $H^k(D, \mathbb{X})$ and $\|\cdot\|_{H^k(D, \mathbb{X})}$. The \mathbb{X} in the subscripts of (semi)norm notation will be omitted if there is no concern of confusion. Furthermore, if D is the domain Ω of our boundary value problem, then we abbreviate $(\cdot, \cdot)_\Omega$ to (\cdot, \cdot) . On matrix-valued functions, the divergence, denoted by div , is calculated row wise. Doing so in the sense

of distributions, we define

$$H(\operatorname{div}, D, \mathbb{M}) := \{\omega \in L^2(D, \mathbb{M}) : \operatorname{div} \omega \in L^2(D, \mathbb{R}^N)\}$$

For a set D in d -dimensional hyperspace ($d \leq N$) and a nonnegative integer k , $\mathcal{P}_k(D)$ is the space of polynomials of degree at most k , restricted to the domain D ; similarly, $\mathcal{P}_k(D, \mathbb{X})$ denotes the space of \mathbb{X} -valued polynomials of degree $\leq k$.

2.2. Elasticity equations. Let $\Omega \subset \mathbb{R}^3$ be a domain with a Lipschitz polyhedral boundary. We denote the spaces of three-dimensional column vectors and the space of 3×3 symmetric matrices by \mathbb{R}^3 and \mathbb{S} . We consider the linear elasticity equations with homogeneous Dirichlet boundary conditions:

$$\begin{aligned} (2.1a) \quad & \mathcal{A}\sigma = \epsilon(u) && \text{in } \Omega, \\ (2.1b) \quad & \operatorname{div} \sigma = f && \text{in } \Omega, \\ (2.1c) \quad & u = 0 && \text{on } \partial\Omega \end{aligned}$$

where $u : \Omega \rightarrow \mathbb{R}^3$ is the displacement vector field, $\sigma : \Omega \rightarrow \mathbb{S}$ is the stress tensor field, $f : \Omega \rightarrow \mathbb{R}^3$ is the load vector, and $\epsilon(u) = \operatorname{sym}(\nabla u)$ is the symmetrized gradient. Here \mathcal{A} is the compliance tensor, a rank-4 tensor field with entries $\mathcal{A}_{ijkl} : \Omega \rightarrow \mathbb{R}$, $1 \leq i, j, k, l \leq 3$ which satisfy symmetry conditions

$$(2.2) \quad \mathcal{A}_{ijkl} = \mathcal{A}_{jikl} = \mathcal{A}_{ijlk} = \mathcal{A}_{klij}.$$

For $\tau \in \mathbb{S}$ with entries τ_{ij} , $1 \leq i, j \leq 3$, $\mathcal{A}(x)\tau$ for $x \in \Omega$ is the matrix whose (i, j) -entry is defined by $\sum_{1 \leq k, l \leq 3} \mathcal{A}_{ijkl}(x)\tau_{kl}$. From (2.2), one can check that $\mathcal{A}(x)\tau$ also belongs to \mathbb{S} , and $\mathcal{A}(x)$ gives a bilinear form on \mathbb{S} by $\mathcal{A}(x)\tau : \omega$ for $\tau, \omega \in \mathbb{S}$ where the colon stands for the Frobenius inner product of matrices. We assume that \mathcal{A} is positive definite over symmetric matrices and is bounded, namely there are positive constants $\theta, \gamma > 0$ such that

$$(2.3) \quad \mathcal{A}(x)\omega : \omega \geq \theta \omega : \omega, \quad \omega \in \mathbb{S}, x \in \Omega,$$

$$(2.4) \quad \|\mathcal{A}\|_{L^\infty(\Omega)} \leq \gamma.$$

In our results for nearly incompressible elastic materials, we make an alternate assumption since (2.3) does not generally hold in such cases: instead of (2.3), we then assume that there exists a $\theta > 0$ such that

$$(2.5) \quad \mathcal{A}(x)\omega : \omega \geq \theta \operatorname{dev}(\omega) : \operatorname{dev}(\omega), \quad \omega \in \mathbb{S}, x \in \Omega$$

where $\operatorname{dev}(\omega) := \omega - \operatorname{tr}(\omega)\mathbb{1}$ with the 3×3 identity matrix $\mathbb{1}$. For isotropic materials, as the Lamé material parameter $\lambda \rightarrow \infty$, we can expect (2.5) to hold, but not (2.3).

A well-known variational formulation of (2.1) is as follows: Find $\sigma \in \Sigma$, $u \in V$ such that

$$(2.6a) \quad (\mathcal{A}\sigma, \tau) + (u, \operatorname{div} \tau) = 0 \quad \text{for all } \tau \in \Sigma,$$

$$(2.6b) \quad (\operatorname{div} \sigma, v) = (f, v) \quad \text{for all } v \in V$$

where $\Sigma = H(\operatorname{div}, \Omega, \mathbb{S})$, $V := [L^2(\Omega)]^3$.

2.3. The finite element. A canonical finite element discretization of (2.6) is to find $\sigma_h \in \Sigma_h$, $u_h \in V_h$ such that

$$(2.7a) \quad (\mathcal{A}\sigma_h, \tau) + (u_h, \operatorname{div} \tau) = 0 \quad \text{for all } \tau \in \Sigma_h,$$

$$(2.7b) \quad (\operatorname{div} \sigma_h, v) = (f, v) \quad \text{for all } v \in V_h$$

with suitable finite element spaces $\Sigma_h \subset \Sigma$, $V_h \subset V$. It is well-known that the finite element pair (Σ_h, V_h) needs to satisfy the Babuška–Brezzi stability conditions for existence and uniqueness of numerical solutions and for accurate approximation of exact solutions (see, e.g., [12]).

To construct a finite element subspace Σ_h , we work on Alfeld simplicial complexes and start by establishing notation associated to an Alfeld split. Starting with a tetrahedron $T = [x_0, x_1, x_2, x_3]$, let T^A be an Alfeld triangulation of T , i.e., we choose an interior point z of T and we let $T_0 = [z, x_1, x_2, x_3]$, $T_1 = [z, x_0, x_2, x_3]$, $T_2 = [z, x_0, x_1, x_3]$, $T_3 = [z, x_0, x_1, x_2]$ and set $T^A = \{T_0, T_1, T_2, T_3\}$.

We let \mathcal{T}_h be a conforming triangulation Ω with tetrahedra and \mathcal{T}_h^A be the resulting mesh after performing an Alfeld split on each $T \in \mathcal{T}_h$. For $T \in \mathcal{T}_h$ we define the local spaces as

$$(2.8) \quad \Sigma_h(T) := \{\omega \in H(\operatorname{div}, T, \mathbb{S}) : \omega|_K \in \mathcal{P}_1(K, \mathbb{S}), \text{ for all } K \in T^A\},$$

$$(2.9) \quad V_h(T) := [\mathcal{P}_1(T)]^3.$$

Although $\operatorname{div}(\Sigma_h(T))$ is not contained in $V_h(T)$, it is contained in

$$W_h(T) := \{v \in [L^2(T)]^3 : v \in [\mathcal{P}_0(K)]^3, \text{ for all } K \in T^A\}.$$

Note that $W_h(T)$ and $V_h(T)$ have the same dimension. We let P_T denote the L^2 -orthogonal projection onto $W_h(T)$:

$$(2.10) \quad (P_T v, w)_T = (v, w)_T \quad \text{for all } w \in W_h(T), \text{ for all } T \in \mathcal{T}_h.$$

For every $K \in T^A$ let x_K be the barycenter of K . We let $I_T : W_h(T) \mapsto V_h(T)$ be defined by

$$(2.11) \quad I_T w(x_K) = w(x_K) \quad \text{for } K \in T^A.$$

A simple quadrature argument gives that

$$(2.12) \quad (I_T w, m)_T = (w, m)_T \quad \text{for all } w, m \in W_h(T),$$

which, in particular, shows that I_T is an injection. Since $\dim W_h(T) = \dim V_h(T)$, injectivity of I_T implies bijectivity of I_T . Moreover, it is easy to see from (2.10) and (2.12) that

$$(2.13) \quad P_T I_T w = w, \quad \text{for all } w \in W_h(T),$$

so the restriction of P_T to $V_h(T)$, namely $P_T|_{V_h(T)} : V_h(T) \rightarrow W_h(T)$, and $I_T : W_h(T) \rightarrow V_h(T)$ are inverses of each other.

The next lemma is crucial for identifying degrees of freedom of the stress element $\Sigma_h(T)$. First we recall some preliminaries that are standard (see e.g. [2]). Let \mathbb{M} denote

the space of 3×3 matrices and let $\Xi : \mathbb{M} \rightarrow \mathbb{M}$ be defined by

$$(2.14) \quad \Xi M = M' - \text{tr}(M)\mathbb{1},$$

where $(\cdot)'$ denotes the transpose. The inverse of this operator is given by $\Xi^{-1}M = M' - \frac{1}{2}\text{tr}(M)\mathbb{1}$. We define $\text{mskw} : \mathbb{R}^3 \rightarrow \mathbb{K}$ and vskw by

$$\text{mskw} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} := \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}, \quad \text{vskw} := \text{mskw}^{-1} \circ \text{skw}.$$

It is easy to see that

$$(2.15) \quad \text{div } \Xi = 2\text{vskw curl}$$

holds. Let $\mathcal{P}_p(T^A) = \{u : u|_K \in \mathcal{P}_p(K) \text{ for all } K \in T^A\}$ and let $\mathring{\mathcal{L}}_p(T^A) = \mathcal{P}_p(T^A) \cap \mathring{H}^1(T)$. Using this notation, we prove the next lemma showing that the space of divergence-free elements with vanishing traces of $\Sigma_h(T)$ is trivial. Our proof of this lemma marks the main point of departure from the analysis of [33]. An alternative to the approach of [33], is to prove this lemma using local discrete elasticity sequences on Alfeld splits obtained by small modifications of the recent three-dimensional exact sequences in [14]. Nonetheless, we have chosen to present a new proof that is easier and generalizable to higher dimensions (as we shall demonstrate in Section 6). Throughout, n denotes the unit outward normal vector field on the boundary of a domain under consideration.

Lemma 2.1. *The equality $\{\omega \in \Sigma_h(T) : \text{div } \omega = 0, \omega n|_{\partial T} = 0\} = \{0\}$ holds.*

Proof. Consider the matrix analogue of the well-known lowest order Brezzi-Douglas-Marini (BDM) space, namely $BDM_1(T^A) = \{\omega \in H(\text{div}, T, \mathbb{M}) : \omega|_K \in \mathcal{P}_1(K, \mathbb{M}) \text{ for all } K \in T^A\}$ and let $\mathring{BDM}_1(T^A) = \{\omega \in BDM_1(T^A) : \omega n = 0 \text{ on } \partial T\}$. Note that

$$\{\omega \in \Sigma_h(T) : \text{div } \omega = 0, \omega n|_{\partial T} = 0\} = \{\omega \in \mathring{BDM}_1(T^A) : \text{vskw } \omega = 0, \text{div } \omega = 0\}.$$

Let $\tilde{\mathcal{P}}_0(T^A) = \{u \in \mathcal{P}_0(T^A) : \int_T u = 0\}$ and let

$$X = \left\{ (u, v) : u \in [\mathcal{P}_1(T^A)]^3, v \in [\tilde{\mathcal{P}}_0(T^A)]^3, \int_T (u \cdot c + v \cdot (c \times x)) = 0 \text{ for all } c \in \mathbb{R}^3 \right\}.$$

The proof is based on the operator $A : \mathring{BDM}_1(T^A) \rightarrow X$ given by

$$(2.16) \quad A\eta = (\text{vskw } \eta, \text{div } \eta), \quad \eta \in \mathring{BDM}_1.$$

Observe that for any $c \in \mathbb{R}^3$ and any $\eta \in \mathring{BDM}_1(T^A)$,

$$(2.17) \quad \int_T c \cdot \text{vskw } \eta = \int_T \eta : \text{mskw } c = \int_T \eta : \text{grad } (c \times x) = - \int_T \text{div } \eta \cdot (c \times x).$$

Thus A indeed maps into X . We proceed to show that A is surjective.

Let $(u, v) \in X$. Since v has components of zero mean, by a standard exact sequence property (e.g. [7]), there exists a $\sigma \in \mathring{BDM}_1(T^A)$ such that

$$(2.18) \quad \text{div } \sigma = v.$$

Since $(u, v) \in X$, this implies

$$0 = \int_T (u \cdot c + v \cdot (c \times x)) = \int_T (u \cdot c + \operatorname{div} \sigma \cdot (c \times x)) = \int_T (u - \operatorname{vskw} \sigma) \cdot c$$

for all $c \in \mathbb{R}^3$, where we have again used (2.17) to obtain the last equality. Hence by a well-known exact sequence property ([41, 28, 25]) on Alfled splits, there exists an $\eta \in \mathring{\mathcal{L}}_2(T^A, \mathbb{M})$ such that

$$(2.19) \quad \operatorname{div} \eta = 2(u - \operatorname{vskw} \sigma).$$

Now $\tau = \sigma + \operatorname{curl} \Xi^{-1} \eta$ is obviously in $B\mathring{D}M_1(T^A)$ and satisfies

$$\operatorname{vskw} \tau = \operatorname{vskw} \sigma + \frac{1}{2} \operatorname{div} \eta = u, \quad (\text{by (2.15) and (2.19)})$$

$$\operatorname{div} \tau = v. \quad (\text{by (2.18)})$$

Thus we have proved that A is surjective.

To conclude, the surjectivity of A implies that $\operatorname{rank}(A) = \dim X \geq 3 \dim \mathcal{P}_1(T^A) + 3 \dim \tilde{\mathcal{P}}_0(T^A) - 3 = 54$. Since $\dim B\mathring{D}M_1(T^A) = 3 \cdot 6 \cdot 3 = 54$ on an Alfled split (see e.g., [14, eq. (2.3)]), the rank-nullity theorem implies that the null space of A is trivial. It means that $\{\omega \in B\mathring{D}M_1 : \operatorname{vskw} \omega = 0, \operatorname{div} \omega = 0\} = \{0\}$, so the conclusion follows. \square

We can now state unisolvent degrees of freedom (“dofs”) for the space $\Sigma_h(T)$. Let $\Delta_k(T)$ denote the set of k -subsimplices of a simplex T , and let $\Delta_k(\mathcal{T}_h)$ be the union of $\Delta_k(T)$ for all $T \in \mathcal{T}_h$.

Theorem 2.2. *The dimension of $\Sigma_h(T)$ is 42. Moreover, an element $\omega \in \Sigma_h(T)$ is uniquely determined by the following dofs:*

$$(2.20a) \quad \int_F \omega n \cdot \kappa, \quad \kappa \in [\mathcal{P}_1(F)]^3, F \in \Delta_2(T), \quad (36 \text{ dofs})$$

$$(2.20b) \quad \int_T \omega. \quad (6 \text{ dofs})$$

Proof. Since $\Sigma_h(T) = \{\omega \in BDM_1(T^A) : \int_T \omega : \eta = 0 \text{ for all } \eta \in \mathcal{P}_1(T^A, \mathbb{K})\}$,

$$\begin{aligned} \dim \Sigma_h(T) &\geq \dim BDM_1(T^A) - \dim \mathcal{P}_1(T^A, \mathbb{K}) \\ &= 90 - 48 = 42, \end{aligned}$$

which is exactly the number of dofs given in (2.20). We now show that $\dim \Sigma_h(T) \leq 42$.

Let $\omega \in \Sigma_h(T)$ for which all dofs of (2.20) vanish. We will show that $\omega = 0$. The dofs (2.20a) give that $\omega n = 0$ on ∂T . Let $v \in \mathcal{P}_1(T)$, then integration by parts gives

$$(2.21) \quad \int_T \operatorname{div} \omega \cdot v = - \int_T \omega : \epsilon(v) = 0$$

where we used (2.20b) in the last equality. Note that $\operatorname{div} \omega \in W_h(T)$. Hence, substituting $v = I_T \operatorname{div} \omega$ in (2.21) and using (2.12) we get $0 = \int_T v \operatorname{div} \omega = \int_T |\operatorname{div} \omega|^2$ which shows that $\operatorname{div} \omega = 0$. By Lemma 2.1, ω vanishes. This proves that $\dim \Sigma_h(T) \leq 42$, so $\dim \Sigma_h(T) = 42$. The argument also shows that the dofs (2.20) uniquely determine an element $\omega \in \Sigma_h(T)$. \square

Unisolvent dofs of a finite element generate an associated canonical interpolant. Let Π_T denote this canonical projection into $\Sigma_h(T)$. It satisfies

$$(2.22a) \quad \int_F \Pi_T \omega n \cdot \kappa = \int_F \omega n \cdot \kappa, \quad \kappa \in [\mathcal{P}_1(F)]^3, F \in \Delta_2(T),$$

$$(2.22b) \quad \int_T \Pi_T \omega = \int_T \omega$$

for any ω in $D_{\Pi_T} := \{\omega \in H(\text{div}, T, \mathbb{S}) : \omega n|_F \in [L^2(F)]^3, \text{ for all } F \in \Delta_2(T)\}$. It follows that

$$(2.23) \quad \int_T \text{div}(\Pi_T \omega - \omega) \cdot v = 0 \quad \text{for all } v \in V_h(T), \omega \in D_{\Pi_T}$$

by integrating by parts and (2.22).

The global discrete spaces are given by

$$(2.24) \quad \Sigma_h := \{\omega \in H(\text{div}, \Omega, \mathbb{S}) : \omega|_T \in \Sigma_h(T), \text{ for all } T \in \mathcal{T}_h\},$$

$$(2.25) \quad V_h := \{v \in [L^2(\Omega)]^3 : v|_T \in V_h(T), \text{ for all } T \in \mathcal{T}_h\},$$

$$(2.26) \quad W_h := \{w \in [L^2(\Omega)]^3 : w|_T \in W_h(T), \text{ for all } T \in \mathcal{T}_h\}.$$

We also define the global interpolant $\Pi : D_{\Pi} \mapsto \Sigma_h$ by $\Pi\omega|_T = \Pi_T\omega$ for all $T \in \mathcal{T}_h$ where

$$(2.27) \quad D_{\Pi} := \{\omega \in H(\text{div}, \Omega, \mathbb{S}) : \omega n|_F \in [L^2(F)]^3, \text{ for all } F \in \Delta_2(\mathcal{T}_h)\}.$$

Note that $H^1(\Omega, \mathbb{S}) \subset D_{\Pi}$. Finally, we need the global versions of P_T and I_T : the projection P onto W_h and $I : W_h \mapsto V_h$ are defined by $Pv|_T = P_Tv$ and $Iw|_T = I_Tw$ for all $T \in \mathcal{T}_h$. The next result is an easy consequence of (2.23).

Proposition 2.3. *For all $v \in V_h$ and $\omega \in D_{\Pi}$,*

$$(2.28) \quad (\text{div}(\Pi\omega - \omega), v) = 0.$$

Moreover, for $s = 1, 2$, for all $\omega \in H^s(\Omega, \mathbb{S})$,

$$(2.29) \quad \|\omega - \Pi\omega\|_{L^2(\Omega)} + h \|\text{div}(\omega - \Pi\omega)\|_{L^2(\Omega)} \leq Ch^s \|\omega\|_{H^s(\Omega)}.$$

The next result follows easily by combining a well-known technique of proving well-posedness [23] with a more recent result on regular potentials [20, Corollary 4.7] that show that given any $z \in H^\ell(\Omega)$, for any real ℓ , there exists a $u \in [H^{\ell+1}(\Omega)]^3$ such that

$$(2.30) \quad \text{div } u = z, \quad \|u\|_{H^{\ell+1}(\Omega)} \leq C \|z\|_{H^\ell(\Omega)}.$$

Hereon, C will denote a generic mesh-independent constant whose value at different occurrences may vary.

Lemma 2.4. *Given any $v \in [L^2(\Omega)]^3$, there exists a $\tau \in H^1(\Omega, \mathbb{S})$ such that*

$$(2.31) \quad \text{div } \tau = v, \quad \|\tau\|_{H^1(\Omega, \mathbb{S})} \leq C \|v\|_{L^2(\Omega)}.$$

Proof. Using (2.30) with $\ell = 0$ for each component of v , we find a $\tau_1 \in H^1(\Omega, \mathbb{M})$ such that

$$(2.32) \quad \operatorname{div} \tau_1 = v, \quad \|\tau_1\|_{H^1(\Omega, \mathbb{M})} \leq C\|v\|_{L^2(\Omega)}.$$

Next, we apply (2.30) with $\ell = 1$ and $z = -2 \operatorname{vskw} \tau_1$ to find a ρ such that $\operatorname{div}(\Xi\rho) = -2 \operatorname{vskw} \tau_1$. Since $\operatorname{vskw} \tau_1 \in [H^1(\Omega)]^3$, and Ξ is a pointwise algebraic bijection, the resulting ρ is in $H^2(\Omega, \mathbb{M})$ and

$$(2.33) \quad \|\rho\|_{H^2(\Omega)} \leq C\|\tau_1\|_{H^1(\Omega)}.$$

Then putting $\tau = \tau_1 + \operatorname{curl} \rho$ we find, using (2.15), that $\operatorname{div} \tau = v$ and

$$2 \operatorname{vskw} \tau = 2 \operatorname{vskw} \tau_1 + 2 \operatorname{vskw}(\operatorname{curl} \rho) = 2 \operatorname{vskw} \tau_1 + \operatorname{div}(\Xi\rho) = 0.$$

Combining (2.32) and (2.33), we conclude that $\tau = \tau_1 + \operatorname{curl} \rho$ is in $H^1(\Omega, \mathbb{S})$, satisfies $\operatorname{div} \tau = v$, as well as the estimate in (2.31). \square

Theorem 2.5 (Discrete stability). *There exists a constant $\beta > 0$ such that*

$$(2.34) \quad \beta\|v\|_{L^2(\Omega)} \leq \sup_{0 \neq \omega \in \Sigma_h} \frac{(\operatorname{div} \omega, v)}{\|\omega\|_{H(\operatorname{div}, \Omega)}} \quad \text{for all } v \in V_h.$$

Proof. Given a $v \in V_h$, we apply Lemma 2.4 to get a τ in $H^1(\Omega, \mathbb{S})$ satisfying (2.31). Since $H^1(\Omega, \mathbb{S}) \subset D_\Pi$, the function $\omega = \Pi\tau$ exists in Σ_h . Then we have

$$(2.35) \quad \|v\|_{L^2(\Omega)}^2 = (\operatorname{div} \tau, v) = (\operatorname{div} \omega, v)$$

by (2.28). The result now follows after using (2.29) with $s = 1$ and (2.31) which gives $\|\omega\|_{H(\operatorname{div}, \Omega)} \leq C\|\tau\|_{H^1(\Omega)} \leq C\|v\|_{L^2(\Omega)}$. \square

The mixed method with Johnson–Mercier element finds $\sigma_h \in \Sigma_h$ and $u_h \in V_h$ satisfying

$$(2.36a) \quad (\mathcal{A}\sigma_h, \tau) + (u_h, \operatorname{div} \tau) = 0 \quad \text{for all } \tau \in \Sigma_h,$$

$$(2.36b) \quad (\operatorname{div} \sigma_h, v) = (f, v) \quad \text{for all } v \in V_h.$$

2.4. Error analysis. We prove error estimates for (2.36). Let $\|\tau\|_{\mathcal{A}} = (\mathcal{A}\tau, \tau)^{1/2}$.

Theorem 2.6. *Let σ, u solve (2.6) and σ_h, u_h solve (2.36). Then, we have*

$$(2.37) \quad \|\Pi\sigma - \sigma_h\|_{\mathcal{A}} \leq \|\Pi\sigma - \sigma\|_{\mathcal{A}}$$

and

$$(2.38) \quad \|u - u_h\|_{L^2(\Omega)} \leq C(\|\sigma - \sigma_h\|_{\mathcal{A}} + \|Qu - u\|_{L^2(\Omega)})$$

where Q is the L^2 -orthogonal projection onto V_h .

Proof. Using (2.28), (2.36b) and (2.6b) we get that

$$(\operatorname{div}(\Pi\sigma - \sigma_h), v) = 0 \quad \text{for all } v \in V_h.$$

However, $(\operatorname{div}(\Pi\sigma - \sigma_h), Pv) = (\operatorname{div}(\Pi\sigma - \sigma_h), v)$. Since $P : V_h \rightarrow W_h$ is an isomorphism, this implies that

$$(2.39) \quad \operatorname{div}(\Pi\sigma - \sigma_h) = 0.$$

Then, (2.37) follows from (2.36a) and (2.6a).

Next, using Lemma 2.4, we obtain a $\tau \in H^1(\Omega, \mathbb{S})$ satisfying

$$(2.40) \quad \operatorname{div} \tau = Qu - u_h,$$

$$(2.41) \quad \|\tau\|_{H^1(\Omega)} \leq C\|Qu - u_h\|_{L^2(\Omega)}.$$

Combining (2.40) with (2.28),

$$\begin{aligned} \|Qu - u_h\|_{L^2(\Omega)}^2 &= (Qu - u_h, Qu - u_h) = (u - u_h, Qu - u_h) \\ &= (u - u_h, \operatorname{div} \Pi\tau) + (u - u_h, \operatorname{div}(\tau - \Pi\tau)) \quad (\text{by (2.40)}) \\ &= -(\mathcal{A}(\sigma - \sigma_h), \Pi\tau) + (u - u_h, \operatorname{div}(\tau - \Pi\tau)) \quad (\text{by (2.6a), (2.36a)}) \\ &= -(\mathcal{A}(\sigma - \sigma_h), \Pi\tau) + (u - Qu, \operatorname{div}(\tau - \Pi\tau)) \quad (\text{by (2.28)}). \end{aligned}$$

By the Cauchy–Schwarz inequality, (2.29), and (2.41),

$$\|Qu - u_h\|_{L^2(\Omega)} \leq C(\|\sigma - \sigma_h\|_{\mathcal{A}} + \|Qu - u\|_{L^2(\Omega)}),$$

so (2.38) now follows by the triangle inequality. \square

Using the above result along with (2.29) and the approximation properties of Q we obtain the following corollary.

Corollary 2.7. *Under the hypothesis of Theorem 2.6, assuming also that $\sigma \in H^2(\Omega, \mathbb{S})$ and $u \in [H^2(\Omega)]^3$, we have*

$$(2.42) \quad \|\sigma - \sigma_h\|_{\mathcal{A}} + \|u - u_h\|_{L^2(\Omega)} \leq Ch^2(\|\sigma\|_{H^2(\Omega)} + \|u\|_{H^2(\Omega)}).$$

2.5. Superconvergence and postprocessing. In this subsection, we prove that a projection of the error $u - u_h$ superconverges at a rate h^3 if we assume full elliptic regularity, i.e., if we assume that Ω and \mathcal{A} are such that the inequality

$$(2.43) \quad \|u\|_{H^2(\Omega)} + \|\sigma\|_{H^1(\Omega)} \leq C\|f\|_{L^2(\Omega)}$$

holds for any solution σ, u of (2.1) obtained with an $f \in L^2(\Omega)$. We will need the following commuting property:

$$(2.44) \quad \operatorname{div}(\Pi\omega - \omega) = 0, \quad \omega \in H^1(\Omega, \mathbb{S}), \operatorname{div} \omega \in W_h.$$

Indeed, for such ω , since $\operatorname{div}(\Pi\omega - \omega) \in W_h$, for all $w \in W_h$, we have $(\operatorname{div}(\Pi\omega - \omega), w) = (\operatorname{div}(\Pi\omega - \omega), Iw)$ by (2.12), and the latter vanishes by (2.28).

Theorem 2.8. *Let σ, u solve (2.6) and σ_h, u_h solve (2.36). Assume that the full regularity estimate (2.43) holds. Then, the following estimate holds*

$$(2.45) \quad \|P(u - u_h)\|_{L^2(\Omega)} \leq Ch\|\sigma - \sigma_h\|_{\mathcal{A}} + Ch^2\|\operatorname{div}(\sigma - \Pi\sigma)\|_{L^2(\Omega)}.$$

If $\sigma \in H^2(\Omega)$, this implies

$$(2.46) \quad \|P(u - u_h)\|_{L^2(\Omega)} \leq Ch^3|\sigma|_{H^2(\Omega)}.$$

Proof. Let τ and ψ solve

$$(2.47a) \quad \mathcal{A}\tau = \epsilon(\psi) \quad \text{in } \Omega,$$

$$(2.47b) \quad \operatorname{div} \tau = P(u - u_h) \quad \text{in } \Omega,$$

$$(2.47c) \quad \psi = 0 \quad \text{on } \partial\Omega.$$

By the full elliptic regularity assumption (2.43), we have

$$(2.48) \quad \|\tau\|_{H^1(\Omega)} + \|\psi\|_{H^2(\Omega)} \leq C\|P(u - u_h)\|_{L^2(\Omega)}.$$

Then,

$$\begin{aligned} \|P(u - u_h)\|_{L^2(\Omega)}^2 &= (P(u - u_h), \operatorname{div} \tau) && \text{(by (2.47b))} \\ &= (P(u - u_h), \operatorname{div} \Pi\tau) && \text{(by (2.44))} \\ &= (u - u_h, \operatorname{div} \Pi\tau) \\ &= -(\mathcal{A}(\sigma - \sigma_h), \Pi\tau) && \text{(by (2.36a), (2.6a))} \\ &= -(\mathcal{A}(\sigma - \sigma_h), \Pi\tau - \tau) - (\mathcal{A}(\sigma - \sigma_h), \tau). \end{aligned}$$

We can now simplify the last term

$$\begin{aligned} (\mathcal{A}(\sigma - \sigma_h), \tau) &= (\sigma - \sigma_h, \mathcal{A}\tau) && \text{(by symmetry of } \mathcal{A}) \\ &= (\sigma - \sigma_h, \epsilon(\psi)) && \text{(by (2.47a))} \\ &= -(\operatorname{div}(\sigma - \sigma_h), \psi) && \text{(integration by parts)} \\ &= -(\operatorname{div}(\sigma - \Pi\sigma), \psi - Q\psi) && \text{(by (2.39), (2.28)).} \end{aligned}$$

Hence, we obtain

$$\|P(u - u_h)\|_{L^2(\Omega)}^2 = -(\mathcal{A}(\sigma - \sigma_h), \Pi\tau - \tau) + (\operatorname{div}(\sigma - \Pi\sigma), \psi - Q\psi).$$

The first estimate (2.45) of the theorem now follows after using (2.29), approximation properties of Q , and (2.48). The second follows from the first and (2.29). \square

We can use the superconvergence result to construct a better approximation to u using a local post-processing scheme. Let $\mathcal{R}(T)$ denote the space of rigid displacements on T . Then, we can decompose $W_h(T)$ as

$$(2.49) \quad W_h(T) = P_T\mathcal{R}(T) + (P_T\mathcal{R}(T))^\perp.$$

where

$$\begin{aligned} P_T\mathcal{R}(T) &:= \{P_T v : v \in \mathcal{R}(T)\} = \{w \in W_h(T) : I_T w \in \mathcal{R}(T)\}, \\ (P_T\mathcal{R}(T))^\perp &:= \{w \in W_h(T) : (w, P_T v)_T = 0, \text{ for all } v \in \mathcal{R}(T)\}. \end{aligned}$$

Define the projection $P_T^{\mathcal{R}} : L^2(T) \rightarrow P_T\mathcal{R}(T)$ by

$$(2.50) \quad (P_T^{\mathcal{R}} q, w)_T = (q, w)_T, \quad \text{for all } w \in P_T\mathcal{R}(T).$$

Recalling that P_T is injective on $V_h(T)$ due to (2.13), we have

$$(2.51a) \quad \|v\|_{L^2(T)} \leq C\|P_T v\|_{L^2(T)} \quad \text{for all } v \in V_h(T),$$

by equivalence of norms on finite dimensional spaces and scaling. Similarly,

$$(2.51b) \quad \|q\|_{L^2(T)} \leq Ch_T \|\epsilon(q)\|_{L^2(T)} + \|P_T^{\mathcal{R}} q\|_{L^2(T)} \quad \text{for all } q \in [\mathcal{P}_2(T)]^3.$$

Indeed, to show that the right hand side of (2.51b) is a norm on $[\mathcal{P}_2(T)]^3$, suppose $\epsilon(q) = 0$ on T . Then q must be a rigid displacement. If furthermore $P_T^{\mathcal{R}} q = 0$, then we see that $P_T q = 0$ and (2.51a) shows that $q = 0$. Together with standard scaling arguments, this establishes (2.51b). Define the quadratic space

$$(2.52) \quad S_h(T) := \{q \in [\mathcal{P}_2(T)]^3 : (q, v)_T = 0, \text{ for all } v \in \mathcal{R}(T)\}.$$

Using it, and following Stenberg [39], we define our post-processed approximation next.

The post-processing system finds $u_h^* \in [\mathcal{P}_2(T)]^3$ satisfying

$$(2.53a) \quad (\epsilon(u_h^*), \epsilon(v))_T = (\mathcal{A}\sigma_h, \epsilon(v))_T \quad \text{for all } v \in S_h(T),$$

$$(2.53b) \quad (u_h^*, w)_T = (P_T u_h, w)_T \quad \text{for all } w \in P_T \mathcal{R}(T),$$

for every $T \in \mathcal{T}_h$, given σ_h, u_h solving (2.36). It is obvious from the definition (2.52) of $S_h(T)$ that $\dim S_h(T) + \dim \mathcal{R}(T) = \dim[\mathcal{P}_2(T)]^3$. Since

$$(2.54) \quad \dim \mathcal{R}(T) = \dim P_T \mathcal{R}(T)$$

by the injectivity of P_T on $V_h(T)$, the system (2.53) is square. Moreover, (2.53) is uniquely solvable: indeed if its right hand side vanishes, then, (2.53a) shows that u_h^* is a rigid displacement, which then together with (2.53b) with (2.51b) implies that u_h^* must vanish.

Next, we show that $u - u_h^*$ converges at a rate higher than that expected for $u - u_h$ from Corollary 2.7. Let Q_2 denote the element-wise L^2 -orthogonal projection onto $[\mathcal{P}_2(T)]^3$.

Theorem 2.9. *Let σ, u solve (2.6) and σ_h, u_h solve (2.36). If u_h^* is given by (2.53), then*

$$\begin{aligned} \|u - u_h^*\|_{L^2(T)} &\leq C(\|u - Q_2 u\|_{L^2(T)} + \|P_T(u_h - u)\|_{L^2(T)}) \\ &\quad + Ch_T(\|\epsilon(u - Q_2 u)\|_{L^2(T)} + \|\mathcal{A}(\sigma - \sigma_h)\|_{L^2(T)}). \end{aligned}$$

Proof. Using (2.53a) and (2.6a) we obtain

$$(2.55) \quad (\epsilon(u_h^* - u), \epsilon(v))_T = (\mathcal{A}(\sigma_h - \sigma), \epsilon(v))_T \quad \text{for all } v \in S_h(T).$$

Since the same equality trivially holds for all $v \in \mathcal{R}(T)$, it actually holds for all $v \in [\mathcal{P}_2(T)]^3$. Hence we may choose $v = u_h^* - Q_2 u$ in $[\mathcal{P}_2(T)]^3$ and manipulate to get

$$(2.56) \quad \|\epsilon(u_h^* - Q_2 u)\|_{L^2(T)} \leq \|\epsilon(u - Q_2 u)\|_{L^2(T)} + \|\mathcal{A}(\sigma - \sigma_h)\|_{L^2(T)}.$$

Using (2.51b) we have

$$(2.57) \quad \|u_h^* - Q_2 u\|_{L^2(T)} \leq Ch_T \|\epsilon(u_h^* - Q_2 u)\|_{L^2(T)} + C \|P_T^{\mathcal{R}}(u_h^* - Q_2 u)\|_{L^2(T)}.$$

It is easy to see that $P_T^{\mathcal{R}} Q_2 u = P_T^{\mathcal{R}} P_T Q_2 u$. Also, (2.53b) implies $P_T^{\mathcal{R}} u_h^* = P_T^{\mathcal{R}} P_T u_h$. Therefore,

$$(2.58) \quad \begin{aligned} \|P_T^{\mathcal{R}}(u_h^* - Q_2 u)\|_{L^2(T)} &\leq \|P_T(u_h - Q_2 u)\|_{L^2(T)} \\ &\leq \|P_T(u_h - u)\|_{L^2(T)} + \|u - Q_2 u\|_{L^2(T)}. \end{aligned}$$

The result now follows by using the estimates of (2.56) and (2.58) within (2.57). \square

Corollary 2.10. *Under the hypotheses of Theorems 2.8 and 2.9, assuming also that $\sigma \in H^2(\Omega, \mathbb{S})$, $u \in [H^3(\Omega)]^3$, we have*

$$\|u - u_h^*\|_{L^2(\Omega)} \leq Ch^3(\|\sigma\|_{H^2(\Omega)} + \|u\|_{H^3(\Omega)}).$$

Proof. Apply (2.45), (2.29), and a standard estimate for projection error of Q_2 , to further bound the estimate for $\|u - u_h^*\|_{L^2(\Omega)}$ given by Theorem 2.9. \square

2.6. Robustness in the incompressible limit. We now investigate the convergence of the method for nearly incompressible isotropic materials. Recall that \mathcal{A} remains bounded in this case but may become close to singular, so methods using \mathcal{A}^{-1} can become problematic. One of the advantages of the mixed method is that it only needs \mathcal{A} . Nonetheless, as $\|\cdot\|_{\mathcal{A}}$ -norm becomes weaker in the incompressible limit, it is natural to ask if σ_h converges in a material-independent norm, a question not answered by Corollary 2.7. To answer this, we now use techniques similar to those in [6] to prove that the L^2 error $\|\sigma - \sigma_h\|_{L^2(\Omega)}$ has an error bound that is robust for nearly incompressible materials satisfying the weaker ellipticity inequality (2.5). In particular, for isotropic materials with $\lambda \rightarrow \infty$, the C below does not depend on λ .

Theorem 2.11. *Assume that only (2.5) holds (instead of (2.3)). Let σ, u solve (2.6) and σ_h, u_h solve (2.36). Then,*

$$(2.59) \quad \|\sigma - \sigma_h\|_{L^2(\Omega)} \leq C(\|\sigma - \sigma_h\|_{\mathcal{A}} + h\|\operatorname{div}(\sigma - \Pi\sigma)\|_{L^2(\Omega)}).$$

Proof. It is evident by setting τ_h to the 3×3 identity matrix $\mathbb{1}$ in (2.6) and (2.36) that $\int_{\Omega} \operatorname{tr}(\sigma - \sigma_h) = 0$. Hence there exists $\phi \in [H_0^1(\Omega)]^3$ (see, e.g., [20]) such that

$$(2.60) \quad \operatorname{div} \phi = \operatorname{tr}(\sigma - \sigma_h), \quad \|\phi\|_{H^1(\Omega)} \leq C\|\operatorname{tr}(\sigma - \sigma_h)\|_{L^2(\Omega)}.$$

Using this,

$$(2.61) \quad \begin{aligned} \|\operatorname{tr}(\sigma - \sigma_h)\|_{L^2(\Omega)}^2 &= (\operatorname{tr}(\sigma - \sigma_h), \operatorname{div} \phi) = (\operatorname{tr}(\sigma - \sigma_h)\mathbb{1}, \epsilon(\phi)) \\ &= 3(\sigma - \sigma_h, \epsilon(\phi)) - 3(\operatorname{dev}(\sigma - \sigma_h), \epsilon(\phi)) \\ &= 3(\operatorname{div}(\sigma - \sigma_h), \phi) - 3(\operatorname{dev}(\sigma - \sigma_h), \epsilon(\phi)). \\ &= 3(\operatorname{div}(\sigma - \Pi\sigma), \phi) - 3(\operatorname{dev}(\sigma - \sigma_h), \epsilon(\phi)), \end{aligned}$$

where we used (2.39) in the last step. To estimate the first term on the final right hand side above, we again use (2.39) (and also (2.28), which is implied by (2.39)) to get

$$(2.62) \quad \begin{aligned} (\operatorname{div}(\sigma - \sigma_h), \phi) &= (\operatorname{div}(\sigma - \Pi\sigma), \phi) = (\operatorname{div}(\sigma - \Pi\sigma), \phi - Q\phi) \\ &\leq Ch\|\operatorname{div}(\sigma - \Pi\sigma)\|_{L^2(\Omega)}\|\phi\|_{H^1(\Omega)} \end{aligned}$$

The last term in (2.61) is easily estimated by (2.5):

$$(\operatorname{dev}(\sigma - \sigma_h), \epsilon(\phi)) \leq C\|\sigma - \sigma_h\|_{\mathcal{A}}\|\phi\|_{H^1(\Omega)}.$$

Together with (2.60), these estimates imply that $\|\operatorname{tr}(\sigma - \sigma_h)\|_{L^2(\Omega)} \leq Ch\|\sigma\|_{H^1(\Omega)}$, so the conclusion follows from $C\|\sigma - \sigma_h\|_{L^2(\Omega)} \leq \|\operatorname{dev}(\sigma - \sigma_h)\|_{L^2(\Omega)} + \|\operatorname{tr}(\sigma - \sigma_h)\|_{L^2(\Omega)}$ and (2.5). \square

Remark 2.12. The above proof extends to other cases (as we shall see in later sections) as long as an analogue of the critical ingredient (2.39) is available. Also note that to obtain convergence rates, it is enough to simply use (2.29) in (2.59).

3. PIECEWISE CONSTANT DISPLACEMENTS ON THE REFINEMENT

In this section we show that we can replace the displacement space V_h with W_h and obtain very similar results. This replacement is attractive since $\operatorname{div} \Sigma_h = W_h$ and the equilibrium equation can then be exactly satisfied for numerical solutions. Also, the stress and displacement elements are implemented with respect to the same refined mesh. However, a price to pay, as we shall see now, is that we are not able to prove an $O(h^3)$ superconvergence result (like in (2.46)) unless the right hand side f belongs to W_h .

We start by proving unisolvency of slightly different dofs (cf. (2.20)) for the same stress space Σ_h . It is useful to note at the outset that by (2.49) and (2.54),

$$\dim(P_T \mathcal{R}(T))^\perp = 12 - 6,$$

so the number of dofs in (2.20b) and (3.1b) both equal six.

Theorem 3.1. *An element $\omega \in \Sigma_h(T)$ is uniquely determined by the following dofs:*

$$(3.1a) \quad \int_F \omega n \cdot \kappa, \quad \kappa \in [\mathcal{P}_1(F)]^3, F \in \Delta_2(T), \quad (36 \text{ dofs})$$

$$(3.1b) \quad \int_T \operatorname{div} \omega \cdot v, \quad v \in (P_T \mathcal{R}(T))^\perp. \quad (6 \text{ dofs})$$

Proof. Let $\omega \in \Sigma_h(T)$ and assume that the dofs in (3.1) vanish. Then we must show that $\omega = 0$. The dofs in (3.1a) show that $\omega n = 0$ on ∂T . Then, by integration by parts, we get $\operatorname{div} \omega \in (P_T \mathcal{R}(T))^\perp$, and (3.1b) shows that $\operatorname{div} \omega = 0$. Then, by Lemma 2.1, $\omega = 0$. \square

The mixed method with piecewise constant displacement finds $\hat{\sigma}_h \in \Sigma_h$ and $\hat{u}_h \in W_h$ satisfying

$$(3.2a) \quad (\mathcal{A}\hat{\sigma}_h, \tau) + (\hat{u}_h, \operatorname{div} \tau) = 0 \quad \text{for all } \tau \in \Sigma_h,$$

$$(3.2b) \quad (\operatorname{div} \hat{\sigma}_h, v) = (f, v) \quad \text{for all } v \in W_h.$$

The next result is analogous to Theorem 2.5, so we omit its proof.

Theorem 3.2. *There exists a constant $\beta > 0$ such that*

$$(3.3) \quad \beta \leq \inf_{0 \neq v \in W_h} \sup_{0 \neq \omega \in \Sigma_h} \frac{(\operatorname{div} \omega, v)}{\|\omega\|_{H(\operatorname{div}, \Omega)} \|v\|_{L^2(\Omega)}}.$$

Our error analysis of the method (3.2) does not proceed as in the previous section, but rather through the following auxiliary problem to find $\tilde{\sigma} \in \Sigma$, $\tilde{u} \in V$ such that

$$(3.4a) \quad (\mathcal{A}\tilde{\sigma}, \tau) + (\tilde{u}, \operatorname{div} \tau) = 0 \quad \text{for all } \tau \in \Sigma,$$

$$(3.4b) \quad (\operatorname{div} \tilde{\sigma}, v) = (Pf, v) \quad \text{for all } v \in V.$$

Now note that $\operatorname{div} \tilde{\sigma} = Pf \in W_h$.

Proposition 3.3. *Let σ, u solve (2.6) and $\tilde{\sigma}, \tilde{u}$ solve (3.4). Then*

$$(3.5) \quad \|\sigma - \tilde{\sigma}\|_{\mathcal{A}} + \|u - \tilde{u}\|_{L^2(\Omega)} \leq Ch\|f - Pf\|_{L^2(\Omega)}.$$

Proof. Equations (2.6) and (3.4) imply

$$(3.6) \quad \|\sigma - \tilde{\sigma}\|_{\mathcal{A}}^2 = (f - Pf, u - \tilde{u}) = (f - Pf, (u - \tilde{u}) - P(u - \tilde{u})).$$

However, using element-wise Poincare inequality (see, e.g., [10]) and then using Korn's first inequality ([16, Theorem 6.3-4]),

$$\|(u - \tilde{u}) - P(u - \tilde{u})\|_{L^2(\Omega)} \leq Ch\|\epsilon(u - \tilde{u})\|_{L^2(\Omega)} = Ch\|\mathcal{A}(\sigma - \tilde{\sigma})\|_{L^2(\Omega)}.$$

Using this in (3.6) after an application of Cauchy-Schwarz inequality, the stated bound for $\|\sigma - \tilde{\sigma}\|_{\mathcal{A}}$ follows. The same bound for $\|u - \tilde{u}\|_{L^2(\Omega)}$ now follows from (3.4a), which implies that $(u - \tilde{u}, \operatorname{div} \tau) = (\mathcal{A}(\tilde{\sigma} - \sigma), \tau)$, after choosing a τ such that $\operatorname{div} \tau = u - \tilde{u}$ by Lemma 2.4. \square

Remark 3.4. At this point, we may choose to proceed using the canonical interpolant J corresponding to the dofs in (3.1). One can prove an analogue of (2.29) for J and also that for any $\omega \in D_{\Pi}$ such that $\operatorname{div} \omega \in W_h$, we have $(\operatorname{div}(J\omega - \omega), w) = 0$ for all $w \in W_h$, which is enough for an error analysis using J . But we pursue the quicker alternative of using the same Π from Section 2.

Theorem 3.5. *Let σ, u solve (2.6) and $\hat{\sigma}_h, \hat{u}_h$ solve (3.2). Then,*

$$(3.7) \quad \|\Pi\sigma - \hat{\sigma}_h\|_{\mathcal{A}} \leq \|\Pi\tilde{\sigma} - \hat{\sigma}_h\|_{\mathcal{A}} + \|\Pi(\tilde{\sigma} - \sigma)\|_{\mathcal{A}}$$

and

$$(3.8) \quad C\|u - \hat{u}_h\|_{L^2(\Omega)} \leq \|\sigma - \hat{\sigma}_h\|_{\mathcal{A}} + \|Pu - u\|_{L^2(\Omega)}.$$

Proof. Subtracting (3.2) from (3.4),

$$(3.9) \quad (\mathcal{A}(\tilde{\sigma} - \hat{\sigma}_h), \tau) + (\tilde{u} - \hat{u}_h, \operatorname{div} \tau) = 0 \quad \text{for all } \tau \in \Sigma_h,$$

$$(3.10) \quad (\operatorname{div}(\tilde{\sigma} - \hat{\sigma}_h), v) = 0 \quad \text{for all } v \in W_h,$$

Since $\operatorname{div}(\tilde{\sigma} - \hat{\sigma}_h)$ is in W_h , equation (3.10) implies

$$(3.11) \quad \operatorname{div}(\tilde{\sigma} - \hat{\sigma}_h) = 0.$$

Also, using (2.44),

$$(3.12) \quad \operatorname{div}(\Pi\tilde{\sigma} - \tilde{\sigma}) = 0.$$

Adding (3.11) and (3.12), we have

$$(3.13) \quad \operatorname{div}(\Pi\tilde{\sigma} - \hat{\sigma}_h) = 0.$$

Hence, setting $\tau = \Pi\tilde{\sigma} - \hat{\sigma}_h$ in (3.9), we immediately obtain

$$\|\Pi\tilde{\sigma} - \hat{\sigma}_h\|_{\mathcal{A}} \leq \|\Pi\tilde{\sigma} - \tilde{\sigma}\|_{\mathcal{A}}.$$

The estimate (3.8) follows the same argument as the proof of (2.38). \square

Corollary 3.6. *Let σ, u solve (2.6), $\tilde{\sigma}, \tilde{u}$ solve (3.4) and $\hat{\sigma}_h, \hat{u}_h$ solve (3.2). If $\sigma, \tilde{\sigma} \in H^2(\Omega, \mathbb{S})$, $u \in [H^1(\Omega)]^3$, then*

$$\|\sigma - \hat{\sigma}_h\|_{\mathcal{A}} + h\|u - \hat{u}_h\|_{L^2(\Omega)} \leq Ch^2(\|\sigma\|_{H^2(\Omega)} + \|\tilde{\sigma}\|_{H^2(\Omega)} + \|f\|_{H^1(\Omega)} + \|u\|_{H^1(\Omega)}).$$

The next intermediate result gives an $O(h^3)$ superconvergence result for $P(\tilde{u} - \hat{u}_h)$. It will help us prove superconvergence of the projection of the error $P(u - \hat{u}_h)$.

Lemma 3.7. *Let $\tilde{\sigma}, \tilde{u}$ solve (3.4) and $\hat{\sigma}_h, \hat{u}_h$ solve (3.2). Assume that the full regularity estimate (2.43) holds. Then, the following estimate holds*

$$(3.14) \quad \|P(\tilde{u} - \hat{u}_h)\|_{L^2(\Omega)} \leq Ch\|\tilde{\sigma} - \hat{\sigma}_h\|_{\mathcal{A}}.$$

Proof. Let τ and ψ solve

$$(3.15a) \quad \mathcal{A}\tau = \epsilon(\psi) \quad \text{in } \Omega,$$

$$(3.15b) \quad \operatorname{div} \tau = P(\tilde{u} - \hat{u}_h) \quad \text{in } \Omega,$$

$$(3.15c) \quad \psi = 0 \quad \text{on } \partial\Omega.$$

Since we are assuming full H^2 -elliptic regularity we have:

$$(3.16) \quad \|\tau\|_{H^1(\Omega)} + \|\psi\|_{H^2(\Omega)} \leq C\|P(\tilde{u} - \hat{u}_h)\|_{L^2(\Omega)}.$$

Then,

$$\begin{aligned} \|P(\tilde{u} - \hat{u}_h)\|_{L^2(\Omega)}^2 &= (P(\tilde{u} - \hat{u}_h), \operatorname{div} \tau) && \text{(by (3.15b))} \\ &= (P(\tilde{u} - \hat{u}_h), \operatorname{div} \Pi\tau) && \text{(by (2.44))} \\ &= (\tilde{u} - \hat{u}_h, \operatorname{div} \Pi\tau) \\ &= -(\mathcal{A}(\tilde{\sigma} - \hat{\sigma}_h), \Pi\tau) && \text{(by (3.9))} \\ &= -(\mathcal{A}(\tilde{\sigma} - \hat{\sigma}_h), \Pi\tau - \tau) - (\mathcal{A}(\tilde{\sigma} - \hat{\sigma}_h), \tau). \end{aligned}$$

We can now simplify the last term

$$\begin{aligned} (\mathcal{A}(\tilde{\sigma} - \hat{\sigma}_h), \tau) &= (\tilde{\sigma} - \hat{\sigma}_h, \mathcal{A}\tau) && \text{(by symmetry of } \mathcal{A}) \\ &= -(\tilde{\sigma} - \hat{\sigma}_h, \epsilon(w)) && \text{(by (3.15a))} \\ &= (\operatorname{div}(\tilde{\sigma} - \hat{\sigma}_h), w), && \text{(by integration by parts)} \end{aligned}$$

which vanishes by (3.11). Hence, we obtain

$$\|P(\tilde{u} - \hat{u}_h)\|_{L^2(\Omega)}^2 = -(\mathcal{A}(\tilde{\sigma} - \hat{\sigma}_h), \Pi\tau - \tau).$$

The result now follows after using (2.29) and (3.16). \square

Corollary 3.8. *Under the same hypothesis of the above theorem we have*

$$(3.17) \quad \|P(u - \hat{u}_h)\|_{L^2(\Omega)} \leq Ch^2(\|\sigma\|_{H^2(\Omega)} + \|\tilde{\sigma}\|_{H^2(\Omega)} + \|f\|_{H^1(\Omega)} + \|u\|_{H^1(\Omega)}).$$

If $f \in W_h$ then

$$(3.18) \quad \|P(u - \hat{u}_h)\|_{L^2(\Omega)} \leq Ch^3(\|\sigma\|_{H^2(\Omega)} + \|\tilde{\sigma}\|_{H^2(\Omega)} + \|f\|_{H^1(\Omega)} + \|u\|_{H^1(\Omega)}).$$

Proof. The estimate (3.17) follows from (3.14) and Theorem 3.5 and Proposition 3.3. If $f \in W_h$ then $u = \tilde{u}$ and $\sigma = \tilde{\sigma}$ hence the result follows also from (3.14) and Theorem 3.5 and Proposition 3.3. \square

Theorem 3.9. *Assume that (2.5). Let σ, u solve (2.6), and $\hat{\sigma}_h, \hat{u}_h$ solve (3.2). Then,*

$$\|\sigma - \hat{\sigma}_h\|_{L^2(\Omega)} \leq C(\|\sigma - \hat{\sigma}_h\|_{\mathcal{A}} + h\|f - Pf\|_{L^2(\Omega)}).$$

Proof. The proof of Theorem 2.11 can be repeated with σ and σ_h replaced by $\tilde{\sigma}$ and $\hat{\sigma}_h$, respectively, since the analogue of the critical ingredient there (see Remark 2.12) is available and given by (3.13). Then we obtain an analogue of (2.59),

$$(3.19) \quad C\|\tilde{\sigma} - \hat{\sigma}_h\|_{L^2(\Omega)} \leq \|\tilde{\sigma} - \hat{\sigma}_h\|_{\mathcal{A}} + h\|\operatorname{div}(\tilde{\sigma} - \Pi\tilde{\sigma})\|_{L^2(\Omega)}.$$

The last term vanishes by (2.44).

To complete the proof, it suffices to estimate $\|\sigma - \tilde{\sigma}\|_{L^2(\Omega)}$. Let $\phi \in [H_0^1(\Omega)]^3$ be as in (2.60) with σ_h replaced by $\tilde{\sigma}$. Then, just as in (2.61), we obtain

$$\|\operatorname{tr}(\sigma - \tilde{\sigma})\|_{L^2(\Omega)} = 3(\operatorname{div}(\sigma - \tilde{\sigma}), \phi) - 3(\operatorname{dev}(\sigma - \tilde{\sigma}), \epsilon(\phi)).$$

Now, $(\operatorname{div}(\sigma - \tilde{\sigma}), \phi) = (f - Pf, \phi - P\phi)$, and by (2.5), $\|\operatorname{dev}(\sigma - \tilde{\sigma})\| \leq C\|\sigma - \tilde{\sigma}\|_{\mathcal{A}}$. Hence we obtain

$$C\|\sigma - \tilde{\sigma}\|_{L^2(\Omega)} \leq \|\sigma - \tilde{\sigma}\|_{\mathcal{A}} + h\|f - Pf\|_{L^2(\Omega)}.$$

The proof is finished by combining this with (3.19). \square

To summarize the results in this section, we have shown (in Theorem 3.5) optimal error estimates for both $\hat{\sigma}_h$ and \hat{u}_h when \hat{u}_h is piecewise constant, superconvergence of order $O(h^2)$ for $P(u - \hat{u}_h)$ for general data f , and superconvergence of order $O(h^3)$ for $P(u - \hat{u}_h)$ when $f \in W_h$ (in Corollary 3.8). We have also shown (in Theorem 3.9) the robustness of the method in the incompressible limit.

4. A REDUCED SPACE PAIR

In this section we present a finite element method obtained by reducing certain judiciously chosen dimensions of the 3D Johnson–Mercier pair. We replace the space V_h by the space of element-wise rigid displacements and reduce Σ_h accordingly. The resulting method has optimal error estimates. It is also robust for nearly incompressible materials (a property that we cannot prove for an even further reduced pair of spaces given in Appendix A). A similar reduced element for the 2D Johnson–Mercier pair was proposed in [15].

On a facet with normal n we let $v_t = n \times (v \times n)$ for vector fields v , while for matrix fields ω , we let $\omega_{nn} = \omega n \cdot n$ and $\omega_{nt} = n \times (\omega n \times n) \equiv (\omega n)_t$. Define

$$\begin{aligned} \Sigma_h^R(T) &:= \{\omega \in \Sigma_h(T) : \operatorname{div} \omega \in P_T \mathcal{R}(T), \omega_{nt} \in \mathcal{R}(F), \text{ for all } F \in \Delta_2(T)\}, \\ V_h^R(T) &:= \mathcal{R}(T). \end{aligned}$$

Here $\mathcal{R}(F)$ is the space of rigid body motions defined on F . We see that there are $18 = 6 + 4 \times 3$ constraints imposed on $\Sigma_h(T)$ to obtain $\Sigma_h^R(T)$. The corresponding global spaces are denoted by Σ_h^R and V_h^R .

Theorem 4.1. *An element $\omega \in \Sigma_h^R(T)$ is uniquely determined by the following dofs:*

$$(4.1a) \quad \int_F \omega_{nn} \kappa, \quad \kappa \in \mathcal{P}_1(F), F \in \Delta_2(T), \quad (12 \text{ dofs})$$

$$(4.1b) \quad \int_F \omega_{nt} \cdot \kappa, \quad \kappa \in \mathcal{R}(F), F \in \Delta_2(T), \quad (12 \text{ dofs}).$$

Proof. We see that $\dim \Sigma_h^R(T) \geq \dim \Sigma_h(T) - 18 = 24$ which is exactly the number of dofs in (4.1). Now suppose that $\omega \in \Sigma_h^R(T)$ and the dofs (4.1a), (4.1b) vanish. Then, obviously, $\omega n = 0$ on ∂T .

By the definition of the reduced stress space, there is a $v \in \mathcal{R}(T)$ such that $P_T v = \operatorname{div} \omega$. Combined with the vanishing trace $\omega n = 0$ on ∂T ,

$$\begin{aligned} \|\operatorname{div} \omega\|_{L^2(T)}^2 &= \int_T \operatorname{div} \omega \cdot P_T v = \int_T \operatorname{div} \omega \cdot v \\ &= \int_{\partial T} \omega n \cdot v = 0. \end{aligned}$$

This shows that $\operatorname{div} \omega = 0$. Then by Lemma 2.1, ω must vanish. \square

The canonical interpolant of the dofs in (4.1), denoted by $\Pi_T^R \omega \in \Sigma_h^R(T)$, satisfies

$$(4.2a) \quad \int_F (\Pi_T^R \omega)_{nn} \kappa = \int_F \omega_{nn} \kappa, \quad \kappa \in \mathcal{P}_1(F), F \in \Delta_2(T),$$

$$(4.2b) \quad \int_F (\Pi_T^R \omega)_{nt} \cdot \kappa = \int_F \omega_{nt} \cdot \kappa \quad \kappa \in \mathcal{R}(F), F \in \Delta_2(T),$$

and we denote the corresponding global projection by Π^R .

Lemma 4.2. *For any $\omega \in D_\Pi$,*

$$(4.3) \quad (\operatorname{div} (\Pi^R \omega - \omega), v) = 0 \quad \text{for all } v \in V_h^R.$$

Proof. Let $v \in \mathcal{R}(T)$. Then,

$$\begin{aligned} (\operatorname{div} (\Pi^R \omega - \omega), v)_T &= \int_{\partial T} (\Pi^R \omega - \omega) n \cdot v \\ &= \int_{\partial T} (\Pi^R \omega - \omega)_{nn} v \cdot n + \int_{\partial T} (\Pi^R \omega - \omega)_{nt} \cdot v_t = 0, \end{aligned}$$

since $v_t \in \mathcal{R}(F)$. \square

The following two results are similar to their previous counterparts and their proofs are omitted.

Proposition 4.3. *For all $\omega \in H^1(\Omega, \mathbb{S})$,*

$$(4.4) \quad \|\omega - \Pi^R \omega\|_{L^2(\Omega)} + h \|\operatorname{div} (\omega - \Pi^R \omega)\|_{L^2(\Omega)} \leq Ch \|\omega\|_{H^1(\Omega)}.$$

Theorem 4.4. *There exists a constant $\beta > 0$ such that*

$$\beta \leq \inf_{0 \neq v \in V_h^R} \sup_{0 \neq \omega \in \Sigma_h^R} \frac{(\operatorname{div} \omega, v)}{\|\omega\|_{H(\operatorname{div}, \Omega)} \|v\|_{L^2(\Omega)}}.$$

The mixed method with the reduced space pair finds $\sigma_h^R \in \Sigma_h^R$ and $u_h^R \in V_h^R$ satisfying

$$(4.5a) \quad (\mathcal{A} \sigma_h^R, \tau) + (u_h^R, \operatorname{div} \tau) = 0 \quad \text{for all } \tau \in \Sigma_h^R,$$

$$(4.5b) \quad (\operatorname{div} \sigma_h^R, v) = (f, v) \quad \text{for all } v \in V_h^R.$$

We present an *a priori* error estimate for this method.

Theorem 4.5. *Let σ, u solve (2.6) and σ_h^R, u_h^R solve (4.5) then the following holds*

$$(4.6) \quad \|\Pi^R \sigma - \sigma_h^R\|_{\mathcal{A}} \leq \|\Pi^R \sigma - \sigma\|_{\mathcal{A}},$$

and

$$(4.7) \quad \|u - u_h^R\|_{L^2(\Omega)} \leq C(\|\sigma - \sigma_h^R\|_{\mathcal{A}} + \|Q^R u - u\|_{L^2(\Omega)})$$

where $Q^R : [L^2(\Omega)]^3 \mapsto V_h^R$ is the L^2 -orthogonal projection onto V_h^R .

Proof. Using (4.3), (4.5b) and (2.6b) we get that

$$(\operatorname{div}(\Pi^R \sigma - \sigma_h^R), v) = 0 \quad \text{for all } v \in V_h^R.$$

However, $(\operatorname{div}(\Pi^R \sigma - \sigma_h^R), Pv) = (\operatorname{div}(\Pi^R \sigma - \sigma_h^R), v)$. Recalling (2.13), since $P_T : \mathcal{R}(T) \rightarrow P_T \mathcal{R}(T)$ is bijection, we conclude that

$$(4.8) \quad \operatorname{div}(\Pi^R \sigma - \sigma_h^R) = 0.$$

Then, (4.6) follows from (4.5a) and (2.6a). To prove (4.7) one follows the same lines as the proof of (2.38). We leave the details to the reader. \square

Corollary 4.6. *Let σ, u solve (2.6) and σ_h^R, u_h^R solve (4.5). If $\sigma \in H^1(\Omega, \mathbb{S})$ and $u \in [H^1(\Omega)]^3$, then*

$$(4.9) \quad \|\sigma - \sigma_h^R\|_{\mathcal{A}} \leq Ch \|\sigma\|_{H^1(\Omega)},$$

$$(4.10) \quad \|u - u_h^R\|_{L^2(\Omega)} \leq Ch(\|\sigma\|_{H^1(\Omega)} + \|u\|_{H^1(\Omega)}).$$

Theorem 4.7. *Assume that only (2.5) holds. Let σ, u solve (2.6) and σ_h^R, u_h^R solve (2.36). Then,*

$$(4.11) \quad \|\sigma - \sigma_h^R\|_{L^2(\Omega)} \leq C(\|\sigma - \sigma_h^R\|_{\mathcal{A}} + h \|\operatorname{div}(\sigma - \Pi^R \sigma)\|_{L^2(\Omega)}).$$

Proof. The proof proceeds along the same lines as that of Theorem 2.11 since the analogue of the critical ingredient mentioned in Remark 2.12 is supplied by (4.8). \square

5. CONNECTION TO STRESS ELEMENTS WITH WEAKLY IMPOSED SYMMETRY

In this section we discuss connection of the methods proposed in Sections 2 and 3 to mixed finite element methods for elasticity using stresses with weakly imposed stress symmetry, i.e., the symmetry of σ is imposed weakly by the additional variational equation

$$\int_{\Omega} \sigma : \eta = 0, \quad \eta \in \Gamma = L^2(\Omega, \mathbb{K})$$

where \mathbb{K} is the space of $N \times N$ skew-symmetric matrices, with $N = 2$ or 3 . Introducing $\rho = \operatorname{grad} u - \epsilon(u)$, the skew-symmetric part of $\operatorname{grad} u$, the alternative mixed formulation finds $\sigma \in \Sigma^w$, $u \in V$, $\rho \in \Gamma$ such that

$$(5.1a) \quad (\bar{\mathcal{A}}\sigma, \tau) + (u, \operatorname{div} \tau) + (\rho, \tau) = 0 \quad \text{for all } \tau \in \Sigma^w,$$

$$(5.1b) \quad (\operatorname{div} \sigma, v) = (f, v) \quad \text{for all } v \in V,$$

$$(5.1c) \quad (\sigma, \eta) = 0 \quad \text{for all } \eta \in \Gamma,$$

where $\Sigma^w := H(\operatorname{div}, \Omega, \mathbb{M})$ and $\bar{\mathcal{A}}$ is an operator which is identical to \mathcal{A} for $L^2(\Omega, \mathbb{S})$ and is bounded and coercive on $L^2(\Omega, \mathbb{K})$. Since $\Sigma_h^w \subset \Sigma^w$ can be a matrix-valued finite element space such that each row is a standard $H(\operatorname{div})$ finite element, it is easier to construct stable mixed methods for (5.1).

Finite element subspaces $\Sigma_h^w \times V_h^w \times \Gamma_h \subset \Sigma^w \times V \times \Gamma$ giving stable mixed methods for (5.1) of the form

$$(5.2a) \quad (\bar{\mathcal{A}}\hat{\sigma}_h^w, \tau) + (\hat{u}_h^w, \operatorname{div} \tau) + (\rho_h, \tau) = 0 \quad \text{for all } \tau \in \Sigma_h^w,$$

$$(5.2b) \quad (\operatorname{div} \hat{\sigma}_h^w, v) = (f, v) \quad \text{for all } v \in V_h^w,$$

$$(5.2c) \quad (\hat{\sigma}_h^w, \eta) = 0 \quad \text{for all } \eta \in \Gamma_h,$$

has long been studied [5, 38, 37, 24, 3, 11, 27]. In [27] it was proved that on meshes where each element T has a Clough–Tocher or Alfeld split, the spaces

$$(5.3a) \quad \Sigma_h^w = \{\omega \in H(\operatorname{div}, \Omega; \mathbb{M}) : \omega|_K \in \mathcal{P}_k(K, \mathbb{M}), \text{ for all } K \in T^A, T \in \mathcal{T}_h\},$$

$$(5.3b) \quad V_h^w = \{v \in [L^2(T)]^3 : v|_K \in [\mathcal{P}_{k-1}(K)]^d, \text{ for all } K \in T^A, T \in \mathcal{T}_h\},$$

$$(5.3c) \quad \Gamma_h = \{\eta \in L^2(T, \mathbb{K}) : \eta|_K \in \mathcal{P}_k(K, \mathbb{K}), \text{ for all } K \in T^A, T \in \mathcal{T}_h\}$$

give stable methods for $k \geq N - 1$ in the N -dimensional elasticity formulation for $N = 2, 3$. Note that the polynomial degrees of Σ_h^w and Γ_h are the same in (5.3). Hence, the numerical solution $\hat{\sigma}_h^w$ of (5.3) is exactly symmetric pointwise. In [35, Subsection 4.2.3], this result was extended to $k \geq 1$ for $N = 3$. Note that when $k = 1$, the space V_h^w in (5.3b) equals W_h . In particular, it was proved in [35] that

$$(5.4) \quad \text{given any } w \in W_h \text{ and } \eta \in \Gamma_h, \text{ there exists } \tau \in \Sigma_h^w \text{ satisfying } \operatorname{div} \tau = w, (\tau, \eta) = \|\eta\|_{L^2(\Omega)}^2, \text{ and } C\|\tau\|_{H(\operatorname{div}, \Omega)} \leq \|w\|_{L^2(\Omega)} + \|\eta\|_{L^2(\Omega)}.$$

The inf-sup stability of (5.3) follows from (5.4).

Proposition 5.1. *The first two components of the unique solution $(\hat{\sigma}_h^w, \hat{u}_h^w, \rho_h)$ of (5.3), when $k = 1$, coincide with the solution $(\hat{\sigma}_h, \hat{u}_h)$ of (3.2).*

Proof. By (5.2c), $\hat{\sigma}_h^w$ is symmetric, so it is in Σ_h , the three-dimensional Johnson–Mercier element. By restricting τ_h to Σ_h , (5.2a) and (5.2b) are exactly the same as (3.2) because $\bar{\mathcal{A}} = \mathcal{A}$ on $H(\operatorname{div}, \Omega, \mathbb{S})$. Therefore, $\hat{\sigma}_h^w = \hat{\sigma}_h$ and $\hat{u}_h^w = \hat{u}_h$ by uniqueness of solutions. \square

Next, consider the case where the displacement is discretized in the space of piecewise linear functions on the unsplit mesh \mathcal{T}_h , namely the space V_h from Section 2, i.e.,

$$(5.5a) \quad (\bar{\mathcal{A}}\sigma_h^w, \tau) + (u_h^w, \operatorname{div} \tau) + (\rho_h, \tau) = 0 \quad \text{for all } \tau \in \Sigma_h^w,$$

$$(5.5b) \quad (\operatorname{div} \sigma_h^w, v) = (f, v) \quad \text{for all } v \in V_h,$$

$$(5.5c) \quad (\sigma_h^w, \eta) = 0 \quad \text{for all } \eta \in \Gamma_h.$$

Proposition 5.2. *The method (5.5) is inf-sup stable and the first two components of its unique solution $(\sigma_h^w, u_h^w, \rho_h) \in \Sigma_h^w \times V_h \times \Gamma_h$ coincide with (σ_h^w, u_h^w) solving (2.36).*

Proof. Given any $(v, \eta) \in V_h \times \Gamma_h$, applying (5.4) to $(Pv, \eta) \in W_h \times \Gamma_h$, we find that there exists $\tau \in \Sigma_h^w$ satisfying

$$(5.6) \quad \operatorname{div} \tau = Pv, \quad (\tau, \eta) = \|\eta\|_{L^2(\Omega)}^2, \quad C\|\tau\|_{H(\operatorname{div}, \Omega)} \leq \|Pv\|_{L^2(\Omega)} + \|\eta\|_{L^2(\Omega)}.$$

Using this τ ,

$$\begin{aligned} \sup_{\omega \in \Sigma_h^w} \frac{(v, \operatorname{div} \omega) + (\eta, \omega)}{\|\omega\|_{H(\operatorname{div}, \Omega)}} &\geq \frac{(v, \operatorname{div} \tau) + (\eta, \tau)}{\|\tau\|_{H(\operatorname{div}, \Omega)}} \geq C(\|Pv\|_{L^2(\Omega)}^2 + \|\eta\|_{L^2(\Omega)}^2)^{1/2} \\ &\geq C(\|v\|_{L^2(\Omega)}^2 + \|\eta\|_{L^2(\Omega)}^2)^{1/2}, \end{aligned}$$

where we have used (2.51a). Thus, (5.5) is inf-sup stable and uniquely solvable. Now we can use the same uniqueness argument as in the proof of Proposition 5.1 to show that (σ_h^w, u_h^w) and the solution of (2.36) are equal. \square

To conclude this section, we have shown that it is possible to compute the solutions of methods in Sections 2 and 3 using completely standard finite element spaces via formulations (5.2)–(5.3) and (5.5). However, we do not know how to arrive at the reduced space $\Sigma_h^R \times V_h^R$ of Section 4 through methods with weakly imposed stress symmetry. The space $\Sigma_h^R(T)$ does not have full $\mathcal{P}_1(K, \mathbb{M})$ polynomials for $K \in T^A$, so finding an appropriate $\Gamma_h^R \subset \Gamma$ which gives exact symmetry of stress tensors and satisfies the inf-sup stability, appears to be nontrivial.

6. THE N -DIMENSIONAL CASE

This section generalizes the Johnson–Mercier element to higher dimensions. To generalize our arguments in Section 2, we need an N -dimensional version of the identity (2.15). This can be found, described in the language of exterior calculus, in [7, 8, 23], using the so-called “BGG resolution” [3, 13, 21]. Nonetheless, we start by giving an elementary and self-contained proof of such an identity using standard vector notation in Subsection 6.1. We then use it to develop the Johnson–Mercier element in \mathbb{R}^N for any $N \geq 2$.

6.1. A commuting diagram of tensor fields for N -dimensional elasticity. Let $\mathbb{V} = \mathbb{R}^N$, let \mathbb{M} denote the space of $N \times N$ matrices, and let $\mathbb{K} = \operatorname{skw}(\mathbb{M})$ denote the space of skew symmetric matrices. We use \otimes to denote the tensor product of vector spaces. Then for (column) vectors $w, v \in \mathbb{V}$, $w \otimes v = wv^t$ equals the outer product. Let e_i denote the standard unit basis of \mathbb{V} , $\mathbf{E}_{ij} = 2 \operatorname{skw}(e_i \otimes e_j) = e_i e_j^t - e_j e_i^t \in \mathbb{K}$. Let $\Lambda(\mathbb{X})$ be C^∞ -smooth \mathbb{X} -valued fields where \mathbb{X} is one of $\mathbb{V}, \mathbb{M}, \mathbb{K}$ or tensor products of these. Henceforth, we use Einstein’s summation convention.

We continue to use the standard divergence operator on vector fields in $\Lambda(\mathbb{V})$ and the (previously used) row-wise divergence operator on matrix fields in $\Lambda(\mathbb{M})$. It is natural to extend div to $\mathbb{K} \otimes \Lambda(\mathbb{V})$ by

$$\operatorname{div}(\eta \otimes w) = \eta \otimes (\operatorname{div} w), \quad \eta \in \mathbb{K}, \quad w \in \Lambda(\mathbb{V}).$$

This motivates the following extension of divergence to the isomorphic space $\Lambda(\mathbb{K} \otimes \mathbb{V})$: writing an element of $\Lambda(\mathbb{K} \otimes \mathbb{V})$ as $b = b_{ijk} \mathbf{E}_{ij} \otimes e_k$ for some scalar component functions

$b_{ijk}(x)$, we define $\text{div} : \Lambda(\mathbb{K} \otimes \mathbb{V}) \rightarrow \Lambda(\mathbb{K})$ by

$$(6.1) \quad \text{div}(b_{ijk}\mathbf{E}_{ij} \otimes e_k) = (\partial_k b_{ijk})\mathbf{E}_{ij}.$$

Next, define a differential operator $\mathbf{d} : \Lambda(\mathbb{K}) \rightarrow \Lambda(\mathbb{V})$ by

$$(6.2) \quad \mathbf{d}(\eta_{ij}\mathbf{E}_{ij}) := \partial_j(\eta_{ij} - \eta_{ji})e_i, \quad \eta = \eta_{ij}\mathbf{E}_{ij} \in \Lambda(\mathbb{K}).$$

It is natural to extend \mathbf{d} to $\mathbb{V} \otimes \Lambda(\mathbb{K})$ by

$$\mathbf{d}(v \otimes \omega) = v \otimes \mathbf{d}\omega, \quad v \in \mathbb{V}, \omega \in \Lambda(\mathbb{K}).$$

Since $\Lambda(\mathbb{V} \otimes \mathbb{K})$ is isomorphic to $\mathbb{V} \otimes \Lambda(\mathbb{K})$, this motivates the definition of the extension of \mathbf{d} in (6.2) to $\mathbf{d} : \Lambda(\mathbb{V} \otimes \mathbb{K}) \rightarrow \Lambda(\mathbb{M})$ by

$$(6.3) \quad \mathbf{d}(a_{ijk}e_i \otimes \mathbf{E}_{jk}) = \partial_k(a_{ijk} - a_{ikj})e_i \otimes e_j, \quad a_{ijk}e_i \otimes \mathbf{E}_{jk} \in \Lambda(\mathbb{V} \otimes \mathbb{K}).$$

Remark 6.1. Alternately, we can arrive at these definitions through a generalized divergence operator [36], Div , that acts on $\Lambda(\mathbb{V} \otimes \mathbb{V})$ and $\Lambda(\mathbb{V} \otimes \mathbb{V} \otimes \mathbb{V})$ by

$$(6.4) \quad \text{Div}(a_{ij}e_i \otimes e_j) = (\partial_j a_{ij})e_i, \quad \text{Div}(a_{ijk}e_i \otimes e_j \otimes e_k) = (\partial_k a_{ijk})e_i \otimes e_j.$$

Since $\mathbb{K} \otimes \mathbb{V}$ and $\mathbb{V} \otimes \mathbb{K}$ are both subspaces of $\mathbb{V} \otimes \mathbb{V} \otimes \mathbb{V}$, the latter defines Div on $\Lambda(\mathbb{K} \otimes \mathbb{V})$ and $\Lambda(\mathbb{V} \otimes \mathbb{K})$. Since

$$\begin{aligned} \eta &= \eta_{ij}\mathbf{E}_{ij} = (\eta_{ij} - \eta_{ji})e_i \otimes e_j && \in \Lambda(\mathbb{K}), \\ a &= a_{ijk}e_i \otimes \mathbf{E}_{jk} = (a_{ijk} - a_{ikj})e_i \otimes e_j \otimes e_k && \in \Lambda(\mathbb{V} \otimes \mathbb{K}), \\ b &= b_{ijk}\mathbf{E}_{ij} \otimes e_k = (b_{ijk} - b_{jik})e_i \otimes e_j \otimes e_k && \in \Lambda(\mathbb{K} \otimes \mathbb{V}), \end{aligned}$$

applying (6.4) we easily see that equations (6.1), (6.2), and (6.3) are the same as

$$\text{div } b = \text{Div } b, \quad \mathbf{d}\eta = \text{Div } \eta, \quad \mathbf{d}a = \text{Div } a,$$

respectively.

With these definitions, we have

$$(6.5) \quad \text{div } \mathbf{d}\eta = 0, \quad \text{for all } \eta \in \Lambda(\mathbb{K}),$$

$$(6.6) \quad \text{div } \mathbf{d}\omega = 0, \quad \text{for all } \omega \in \Lambda(\mathbb{V} \otimes \mathbb{K}).$$

Indeed, to see that (6.5) holds, it suffices to observe that $\text{div } \mathbf{d}\eta = \text{div}(\partial_j(\eta_{ij} - \eta_{ji})e_i) = \partial_i \partial_j(\eta_{ij} - \eta_{ji}) = 0$. Essentially the same argument also shows (6.6).

Remark 6.2. In the language of exterior calculus, (6.5) and (6.6) imply that the sequences

$$(6.7) \quad \Lambda(\mathbb{K}) \xrightarrow{\mathbf{d}} \Lambda(\mathbb{V}) \xrightarrow{\text{div}} \Lambda(\mathbb{R}),$$

$$(6.8) \quad \Lambda(\mathbb{V} \otimes \mathbb{K}) \xrightarrow{\mathbf{d}} \Lambda(\mathbb{M}) \xrightarrow{\text{div}} \Lambda(\mathbb{V}),$$

form complexes, elucidating the connection to relevant N -dimensional de Rham complexes. Note that the tensor products of \mathbb{V} with function spaces in (6.7) yield their analogue in (6.8) up to an isomorphism.

Next, we define a linear operator $\Theta : \Lambda(\mathbb{V} \otimes \mathbb{K}) \rightarrow \Lambda(\mathbb{K} \otimes \mathbb{V})$ by defining Θa for $a = a_{ijk}e_i \otimes e_j \otimes e_k$ in the larger set $\Lambda(\mathbb{V} \otimes \mathbb{V} \otimes \mathbb{V})$, namely

$$(6.9) \quad \Theta(a_{ijk}e_i \otimes e_j \otimes e_k) := (a_{ijk} - a_{jik}) e_i \otimes e_j \otimes e_k$$

When a is in $\Lambda(\mathbb{V} \otimes \mathbb{K})$, we have $a = a_{ijk}e_i \otimes e_j \otimes e_k = \frac{1}{2}a_{ijk}e_i \otimes \mathbf{E}_{jk}$. Hence, (6.9) can equivalently be expressed as

$$(6.10) \quad \Theta(a_{ijk}e_i \otimes \mathbf{E}_{jk}) = 2a_{ijk}\mathbf{E}_{ij} \otimes e_k.$$

Lemma 6.3. *The operator $\Theta : \Lambda(\mathbb{V} \otimes \mathbb{K}) \rightarrow \Lambda(\mathbb{K} \otimes \mathbb{V})$ is invertible and its inverse is given by*

$$\Theta^{-1}b = \frac{1}{2}(b_{ijk} - b_{ikj} - b_{jki}) e_i \otimes e_j \otimes e_k$$

for any $b = b_{ijk} e_i \otimes e_j \otimes e_k \in \Lambda(\mathbb{K} \otimes \mathbb{V})$. Moreover, for all $\omega \in \Lambda(\mathbb{V} \otimes \mathbb{K})$,

$$(6.11) \quad \operatorname{div} \Theta \omega = 2 \operatorname{skw} \mathbf{d} \omega.$$

Proof. By direct calculation, it is easy to verify that the given expression for Θ^{-1} satisfies $\Theta(\Theta^{-1}b) = b$ for all $b \in \Lambda(\mathbb{K} \otimes \mathbb{V})$ and $\Theta^{-1}(\Theta a) = a$ for all $a \in \Lambda(\mathbb{V} \otimes \mathbb{K})$. To detail the latter, letting $\theta = \Theta a = (a_{ijk} - a_{jik})e_i \otimes e_j \otimes e_k$, observe that

$$\begin{aligned} 2\Theta^{-1}\Theta a &= (\theta_{ijk} - \theta_{ikj} - \theta_{jki})e_i \otimes e_j \otimes e_k \\ &= ((a_{ijk} - a_{jik}) - (a_{ikj} - a_{kij}) - (a_{jki} - a_{kji}))e_i \otimes e_j \otimes e_k \end{aligned}$$

When $a \in \Lambda(\mathbb{V} \otimes \mathbb{K})$, its components have skew symmetry in the last two indices (i.e., $a_{ijk} = -a_{ikj}$), so the last expression above simplifies to $2a_{ijk}e_i \otimes e_j \otimes e_k = 2a$.

Next, to prove (6.11), let $\omega = \omega_{ijk}e_i \otimes \mathbf{E}_{jk} \in \Lambda(\mathbb{V} \otimes \mathbb{K})$. Then, by (6.1) and (6.10),

$$\operatorname{div}(\Theta \omega) = \operatorname{div}(2\omega_{ijk}\mathbf{E}_{ij} \otimes e_k) = 2(\partial_k \omega_{ijk}) \mathbf{E}_{ij}.$$

Also, by (6.3),

$$\begin{aligned} 2 \operatorname{skw} \mathbf{d} \omega &= \partial_k (\omega_{ijk} - \omega_{ikj}) 2 \operatorname{skw}(e_i \otimes e_j) \\ &= \partial_k (\omega_{ijk} - \omega_{jik}) \mathbf{E}_{ij} = 2(\partial_k \omega_{ijk}) \mathbf{E}_{ij}. \end{aligned}$$

Thus the left and right hand sides of (6.11) are equal. \square

In other words, we have shown that the following diagram commutes and that its top and bottom rows are complexes.

$$\begin{array}{ccccc} & & \Lambda(\mathbb{K} \otimes \mathbb{V}) & \xrightarrow{\operatorname{div}} & \Lambda(\mathbb{K}) \\ & \nearrow \Theta & & \nearrow 2\operatorname{skw} & \\ \Lambda(\mathbb{V} \otimes \mathbb{K}) & \xrightarrow{\mathbf{d}} & \Lambda(\mathbb{M}) & \xrightarrow{\operatorname{div}} & \Lambda(\mathbb{V}). \end{array}$$

For later use, we state the following result on composition of trace with \mathbf{d} on a hyperplane.

Lemma 6.4. *Let $\eta = \eta_{ij}\mathbf{E}_{ij} \in \Lambda(\mathbb{K})$ and suppose that Γ is a hyperplane in which the components of η vanish then $\mathbf{d}\eta \cdot n$ vanishes on Γ where n is a unit normal to Γ .*

Proof. After possibly rotating coordinates we can assume without loss of generality that $n = e_N$. Then,

$$d\eta \cdot n = \partial_j(\eta_{ij} - \eta_{ji})e_i \cdot e_N = \partial_j(\eta_{Nj} - \eta_{jN}) = \sum_{j=1}^{N-1} \partial_j(\eta_{Nj} - \eta_{jN}).$$

The latter vanish on Γ since $\eta_{Nj} - \eta_{jN}$ vanish on Γ and ∂_j (for $1 \leq j \leq N-1$) are tangential derivatives on Γ . \square

Of course, we can define the weak version of \mathbf{d} and extend the domain of \mathbf{d} . To this end, let $\eta = \eta_{ij}\mathbf{E}_{ij}$ we say $\mathbf{d}\eta$ exists if there exists $\omega \in L^1(\Omega, \mathbb{V})$ such that

$$(6.12) \quad (\omega, \phi_k e_k) = -(\eta_{ij} - \eta_{ji}, \partial_j \phi_k) \delta_{ik}, \quad \text{for all } \phi = \phi_k e_k \in C_0^\infty(\Omega, \mathbb{V}),$$

and we set $\mathbf{d}\eta = \omega$. Here δ_{ik} is the Kronecker delta. We then define $H(\mathbf{d}, \Omega) := \{\eta \in L^2(\Omega, \mathbb{K}) : \mathbf{d}\eta \in L^2(\Omega, \mathbb{V})\}$. The identities (6.6) and (6.11) hold for $\omega \in \mathbb{V} \otimes H(\mathbf{d}, \Omega)$.

6.2. The stress element on the Alfeld split in higher dimensions. For an N -simplex $T = [x_0, x_1, x_2, \dots, x_N]$, let T^A be an Alfeld split of T , i.e., choosing an interior point z of T , define $T^A = \{T_0, T_1, T_2, \dots, T_N\}$ with N -simplices $T_i = [z, x_0, \dots, \hat{x}_i, \dots, x_N]$ where \hat{x}_i means that x_i is not in the vertices of T_i . For a given triangulation \mathcal{T}_h of Ω , we let \mathcal{T}_h^A be the resulting triangulation after performing an Alfeld split to each $T \in \mathcal{T}_h$. On each macro element $T \in \mathcal{T}_h$, define the local spaces

$$\begin{aligned} \mathcal{P}_k(T^A, \mathbb{X}) &:= \{\omega \in L^2(T, \mathbb{X}) : \omega|_K \in \mathcal{P}_k(K, \mathbb{X}), \text{ for all } K \in T^A\}, \\ \Sigma_h(T) &:= H(\text{div}, T, \mathbb{S}) \cap \mathcal{P}_1(T^A, \mathbb{S}), \\ V_h(T) &:= \mathcal{P}_1(T, \mathbb{V}). \end{aligned}$$

The N -dimensional matrix analogue of the well-known BDM space and its subspace with vanishing normal components on ∂T are namely

$$\begin{aligned} BDM_1(T^A) &:= H(\text{div}, T, \mathbb{M}) \cap \mathcal{P}_1(T^A, \mathbb{M}), \\ \mathring{BDM}_1(T^A) &:= \{\tau \in BDM_1(T^A) : \tau n|_{\partial T} = 0\}. \end{aligned}$$

Notice that the Alfeld split of T has $N+1$ distinct sub-simplices in $\Delta_{N-1}(T)$ and $\frac{1}{2}N(N+1)$ distinct internal subsimplices in $\Delta_{N-1}(T^A)$, therefore $\dim BDM_1(T^A) = (N+1) \cdot N^2 + \frac{1}{2}N(N+1) \cdot N^2 = (\frac{N}{2}+1)(N+1)N^2$ and $\dim \mathring{BDM}_1(T^A) = \frac{1}{2}(N+1)N^3$. Notice also that $\dim \mathcal{P}_1(T^A, \mathbb{K}) = \frac{1}{2}N(N-1)(N+1)^2$.

We will also need the space $\mathring{\mathcal{L}}_2(T^A) := \mathcal{P}_2(T^A) \cap \mathring{H}^1(T)$ and \mathbb{X} -valued versions $\mathring{\mathcal{L}}_2(T^A, \mathbb{X}) = \mathring{\mathcal{L}}_2(T^A) \otimes \mathbb{X}$. We begin by recalling the next result from [25, Theorem 3.1] (see also [28]).

Proposition 6.5. *Let $v \in \mathcal{P}_1(T^A)$ with $\int_T v = 0$ there exists $\rho \in \mathring{\mathcal{L}}_2(T^A, \mathbb{V})$ such that*

$$\text{div } \rho = v.$$

Proposition 6.6. *If $\rho \in \mathcal{L}_2(T^A, \mathbb{V} \otimes \mathbb{K})$ then $\mathbf{d}\rho \in BDM_1(T^A)$. Moreover, if $\rho \in \mathring{\mathcal{L}}_2(T^A, \mathbb{V} \otimes \mathbb{K})$ then $\mathbf{d}\rho \in \mathring{BDM}_1(T^A)$.*

Proof. Let $\rho \in \mathcal{L}_2(T^A, \mathbb{V} \otimes \mathbb{K})$ then for each $K \in T^A$ $\mathbf{d}\rho|_K$ has linear components. Hence, it is enough to show that $\rho \in H(\mathbf{d}, T) \otimes \mathbb{V}$, but this follows immediately from (6.12). If $\rho \in \mathring{\mathcal{L}}_2(T^A, \mathbb{V} \otimes \mathbb{K})$ then we must also show that $(\mathbf{d}\rho)n$ vanishes on ∂T , but this follows from Lemma 6.4. \square

Lemma 6.7. *The equality $\{\omega \in \Sigma_h(T) : \operatorname{div} \omega = 0, \omega n|_{\partial T} = 0\} = \{0\}$ continues to hold in the N -dimensional case.*

Proof. Let $\tilde{\mathcal{P}}_0(T^A) = \{u \in \mathcal{P}_0(T^A) : \int_T u = 0\}$ and set

$$X = \left\{ (\eta, v) : \eta \in \mathcal{P}_1(T^A, \mathbb{K}), v \in \tilde{\mathcal{P}}_0(T^A, \mathbb{V}), \int_T (\eta : \kappa + v \cdot \kappa x) = 0 \text{ for all } \kappa \in \mathbb{K} \right\}$$

where κx is the matrix-vector product with the coordinate vector polynomial x . The proof is based on the operator $A : \mathring{BDM}_1(T^A) \rightarrow X$ given by

$$(6.13) \quad A\tau = (-\operatorname{skw} \tau, \operatorname{div} \tau), \quad \tau \in \mathring{BDM}_1(T^A).$$

Observe that for any $\kappa \in \mathbb{K}$ and $\tau \in \mathring{BDM}_1(T^A)$,

$$(6.14) \quad \int_T \kappa : \operatorname{skw} \tau = \int_T \operatorname{grad}(\kappa x) : \tau = - \int_T (\kappa x) \cdot \operatorname{div} \tau.$$

Thus A indeed maps into X . We proceed to show that A is surjective.

Let $(\eta, v) \in X$. Since v has components of zero mean, by a standard exact sequence property [4], there exists a $\sigma \in \mathring{BDM}_1(T^A)$ such that

$$(6.15) \quad \operatorname{div} \sigma = v.$$

Since $(\eta, v) \in X$, this implies

$$\int_T (\eta - \operatorname{skw} \sigma) : \kappa = \int_T (\eta : \kappa + (\operatorname{div} \sigma) \cdot \kappa x) = \int_T (\eta : \kappa + v \cdot \kappa x) = 0,$$

where we have again used (6.14). Letting $\zeta = \eta - \operatorname{skw} \sigma \in \mathcal{P}_1(T^A, \mathbb{K})$, the above shows all the components have zero mean. Hence, by Proposition 6.5 there exists $\omega \in \mathring{\mathcal{L}}_2(T^A, \mathbb{V} \otimes \mathbb{K})$ such that

$$(6.16) \quad \operatorname{div} \omega = \zeta - \operatorname{skw} \sigma.$$

We set $a = \Theta^{-1}\omega$ and note that $a \in \mathring{\mathcal{L}}_2(T^A, \mathbb{V} \otimes \mathbb{K})$. Setting $\tau = \sigma + \mathbf{d}a$, then by Proposition 6.6 we have $\tau \in \mathring{BDM}_1(T^A)$.

By the commuting diagram property (6.11), $\operatorname{skw} \mathbf{d}a = \operatorname{div} \Theta a = \operatorname{div} \omega$, so we have

$$\begin{aligned} \operatorname{skw} \tau &= \operatorname{skw} \sigma + \operatorname{div} \omega = \zeta, & (\text{by (6.16)}) \\ \operatorname{div} \tau &= v. & (\text{by (6.15) and (6.6)}) \end{aligned}$$

Thus we have proved that A is surjective.

To conclude, the surjectivity of A implies that

$$\begin{aligned} \text{rank}(A) &\geq \dim \mathcal{P}_1(T^A, \mathbb{K}) + \dim \tilde{\mathcal{P}}_0(T^A, \mathbb{V}) - \dim \mathbb{K} \\ &= \frac{1}{2}N(N-1)(N+1)^2 + N^2 - \frac{1}{2}N(N-1) \\ &= \frac{1}{2}N^3(N+1). \end{aligned}$$

Since $\dim BDM_1(T^A) = \frac{1}{2}N^3(N+1)$, the rank-nullity theorem implies that the null space of A is trivial, i.e., $\{\tau \in BDM_1(T^A) : \text{skw } \tau = 0, \text{div } \tau = 0\} = \{0\}$. \square

We now prove the main result of this section. The proof is a straightforward generalization of the proof of Theorem 2.2.

Theorem 6.8. *The dimension of $\Sigma_h(T)$ is $(N + \frac{1}{2})N(N + 1)$. Moreover, an element $\omega \in \Sigma_h(T)$ is uniquely determined by the following dofs:*

$$(6.17a) \quad \int_F \omega n_T \cdot \kappa, \quad \kappa \in \mathcal{P}_1(F, \mathbb{V}), \quad F \in \Delta_{N-1}(T), \quad ((N+1)N^2 \text{ dofs})$$

$$(6.17b) \quad \int_T \omega. \quad (\frac{1}{2}N(N+1) \text{ dofs})$$

Proof. Since $\Sigma_h(T) = \{\omega \in BDM_1(T^A) : \int_T \omega : \eta = 0 \text{ for all } \eta \in \mathcal{P}_1(T^A, \mathbb{K})\}$,

$$\begin{aligned} \dim \Sigma_h(T) &\geq \dim BDM_1(T^A) - \dim \mathcal{P}_1(T^A, \mathbb{K}) \\ &= (\frac{N}{2} + 1)(N+1)N^2 - \frac{N}{2}(N-1)(N+1)^2 \\ &= \frac{N}{2}(N+1)(2N+1). \end{aligned}$$

To show $\dim \Sigma_h(T) = \frac{N}{2}(N+1)(2N+1)$, it is sufficient to prove that $\omega = 0$ if the dofs (6.17) of $\omega \in \Sigma_h(T)$ vanish because the number of dofs in (6.17) is $\frac{N}{2}(N+1)(2N+1)$. This also proves that $\Sigma_h(T)$ is unisolvent by (6.17).

To this end, let $\omega \in \Sigma_h(T)$ have vanishing dofs (6.17). Then, $\omega n = 0$ on ∂T by (6.17a). Moreover, using (6.17b) and integration by parts gives

$$(6.18) \quad \int_T \text{div } \omega \cdot v = - \int_T \omega : \epsilon(v) = 0 \quad \text{for all } v \in \mathcal{P}_1(T, \mathbb{V}).$$

By a simple argument using quadrature rules as in the 3D case, we conclude that $\text{div } \omega = 0$. Hence, Lemma 6.7 implies that $\omega = 0$. \square

APPENDIX A. ANOTHER REDUCED SPACE PAIR

In this section we reduce the spaces even further from the reduced pair in Section 4. In this further reduced finite element method, V_h is replaced by only piecewise constant elements on $T \in \mathcal{T}_h$ and Σ_h is also further reduced correspondingly. We prove optimal \mathcal{A} -weighted L^2 error estimate for σ and L^2 error estimate for u . However, for this further reduced method, we are not able to prove (an analogue of Theorem 2.11 giving)

a robust error estimate for the nearly incompressible case. The reason for this is that we now only have very weak control of the divergence of the stresses. Hence we are unable to recommend this method. Nonetheless, we present it as a curiosity, since it has only three dofs per facet and yet remains stable.

The discrete spaces for this method are

$$\begin{aligned}\tilde{\Sigma}_h^R(T) &:= \{\omega \in \Sigma_h(T) : \operatorname{div} \omega \in P_T \mathcal{R}(T), \omega n \in [\mathcal{P}_0(F)]^3, \text{ for all } F \in \Delta_2(T)\}, \\ \tilde{V}_h^R(T) &:= [\mathcal{P}_0(T)]^3.\end{aligned}$$

The corresponding global spaces are denoted by $\tilde{\Sigma}_h^R, \tilde{V}_h^R$.

Theorem A.1. *An element $\omega \in \tilde{\Sigma}_h^R(T)$ is uniquely determined by the following dofs:*

$$(A.1) \quad \int_F \omega n \cdot \kappa, \quad \kappa \in [\mathcal{P}_0(F)]^3, F \in \Delta_2(T), \quad (12 \text{ dofs}).$$

Proof. We see that $\dim \tilde{\Sigma}_h^R(T) \geq \dim \Sigma_h(T) - (6 + 6 \cdot 4) = 12$ which is exactly the number of dofs in (A.1). Now suppose that $\omega \in \tilde{\Sigma}_h^R(T)$ and the dofs (A.1) vanish. Then, we have $\omega n = 0$ on ∂T .

Let $v \in \mathcal{R}(T)$ be such that $P_T v = \operatorname{div} \omega$ then

$$\|\operatorname{div} \omega\|_{L^2(T)}^2 = \int_T \operatorname{div} \omega \cdot P_T v = \int_T \operatorname{div} \omega \cdot v = \int_{\partial T} \omega n \cdot v = 0.$$

Hence, this shows that $\operatorname{div} \omega = 0$. By Lemma 2.1 we have that ω vanishes. \square

The corresponding projection with the degrees of freedom is defined as follows.

$$(A.2a) \quad \int_F (\tilde{\Pi}_T^R \omega n) \cdot \kappa = \int_F \omega n \cdot \kappa, \quad \kappa \in [\mathcal{P}_0(F)]^3, F \in \Delta_2(T).$$

From this we define the global projection $\tilde{\Pi}^R$.

Lemma A.2. *It holds,*

$$(A.3) \quad (\operatorname{div} (\tilde{\Pi}^R \omega - \omega), v) = 0 \quad \text{for all } v \in \tilde{V}_h^R, \omega \in D_\Pi.$$

Proof. Let $v \in [\mathcal{P}_0(T)]^3$. Then,

$$(\operatorname{div} (\tilde{\Pi}^R \omega - \omega), v)_T = \int_{\partial T} (\tilde{\Pi}^R \omega - \omega) n \cdot v = 0.$$

\square

We omit the proofs of the following results analogous to previous sections.

Proposition A.3. *For all $\omega \in H^1(\Omega, \mathbb{S})$,*

$$(A.4) \quad \|\omega - \tilde{\Pi}^R \omega\|_{L^2(\Omega)} + h \|\operatorname{div} (\omega - \tilde{\Pi}^R \omega)\|_{L^2(\Omega)} \leq Ch \|\omega\|_{H^1(\Omega)}.$$

Theorem A.4. *There exists a constant $\beta > 0$ such that*

$$(A.5) \quad \beta \leq \inf_{0 \neq v \in \tilde{V}_h^R} \sup_{0 \neq \omega \in \tilde{\Sigma}_h^R} \frac{(\operatorname{div} \omega, v)}{\|\omega\|_{H(\operatorname{div}, \Omega)} \|v\|_{L^2(\Omega)}}.$$

The mixed method with the second reduced element finds $\tilde{\sigma}_h^R \in \tilde{\Sigma}_h^R$ and $\tilde{u}_h^R \in \tilde{V}_h^R$ satisfying

$$(A.6a) \quad (\mathcal{A}\tilde{\sigma}_h^R, \tau) + (\tilde{u}_h^R, \operatorname{div} \tau) = 0 \quad \text{for all } \tau \in \tilde{\Sigma}_h^R,$$

$$(A.6b) \quad (\operatorname{div} \tilde{\sigma}_h^R, v) = (f, v) \quad \text{for all } v \in \tilde{V}_h^R.$$

The error analysis of this method is more involved than the previous cases since $\operatorname{div}(\tilde{\Pi}^R \sigma - \tilde{\sigma}_h^R)$ is not necessarily zero. We proceed by proving a stability result for the method, as in [7], using the bilinear form

$$B(\omega_h, v_h; \tau_h, w_h) := (\mathcal{A}\omega_h, \tau_h) + (v_h, \operatorname{div} \tau_h) - (\operatorname{div} \omega_h, w_h).$$

Theorem A.5. *There exists constants c_1 and c_2 such that for every $\omega_h \in \tilde{\Sigma}_h^R$ and $v_h \in \tilde{V}_h^R$, there exists $\tau_h \in \tilde{\Sigma}_h^R$ and $w_h \in \tilde{V}_h^R$, satisfying*

$$(A.7) \quad \|\omega_h\|_{\mathcal{A}}^2 + \|\tilde{P}^R \operatorname{div} \omega_h\|_{L^2(\Omega)}^2 + \|v_h\|_{L^2(\Omega)}^2 \leq c_1 B(\omega_h, v_h; \tau_h, w_h),$$

and

$$(A.8) \quad \begin{aligned} & \|\tau_h\|_{\mathcal{A}}^2 + \|\tilde{P}^R \operatorname{div} \tau_h\|_{L^2(\Omega)}^2 + \|w_h\|_{L^2(\Omega)}^2 \\ & \leq c_2 (\|\omega_h\|_{\mathcal{A}}^2 + \|\tilde{P}^R \operatorname{div} \omega_h\|_{L^2(\Omega)}^2 + \|v_h\|_{L^2(\Omega)}^2). \end{aligned}$$

Proof. By the proof of Theorem A there exists $\rho_h \in \tilde{\Sigma}_h^R$ such that

$$(A.9) \quad \|v_h\|_{L^2(\Omega)}^2 = (\operatorname{div} \rho_h, v_h),$$

$$(A.10) \quad \|\rho_h\|_{H(\operatorname{div}, \Omega)} \leq \kappa \|v_h\|_{L^2(\Omega)}.$$

Choose $\tau_h = \omega_h + \frac{1}{\gamma\kappa^2} \rho_h$ and $w_h = v_h - \tilde{P}^R \operatorname{div} \omega_h$. Then, we have

$$\begin{aligned} B(\omega_h, v_h; \tau_h, w_h) &= \|\omega_h\|_{\mathcal{A}}^2 + \frac{1}{\gamma\kappa^2} (\mathcal{A}\omega_h, \rho_h) + (v_h, \operatorname{div} \omega_h) + \frac{1}{\gamma\kappa^2} (v_h, \operatorname{div} \rho_h) \\ &\quad - (\operatorname{div} \omega_h, v_h) + (\operatorname{div} \omega_h, \tilde{P}^R \operatorname{div} \omega_h) \\ &= \|\omega_h\|_{\mathcal{A}}^2 + \|\tilde{P}^R \operatorname{div} \omega_h\|_{L^2(\Omega)}^2 + \frac{1}{\gamma\kappa^2} \|v_h\|_{L^2(\Omega)}^2 + \frac{1}{\gamma\kappa^2} (\mathcal{A}\omega_h, \rho_h), \end{aligned}$$

where we used (A.9). On the other hand, by (A.10)

$$\begin{aligned} \frac{1}{\gamma\kappa^2} (\mathcal{A}\omega_h, \rho_h) &\geq -\frac{1}{2} \|\omega_h\|_{\mathcal{A}}^2 - \frac{1}{2\gamma^2\kappa^4} \|\rho_h\|_{\mathcal{A}}^2 \\ &\geq -\frac{1}{2} \|\omega_h\|_{\mathcal{A}}^2 - \frac{1}{2\gamma\kappa^2} \|v_h\|_{L^2(\Omega)}^2. \end{aligned}$$

Hence,

$$\frac{1}{2} \|\omega_h\|_{\mathcal{A}}^2 + \|\tilde{P}^R \operatorname{div} \omega_h\|_{L^2(\Omega)}^2 + \frac{1}{2\gamma\kappa^2} \|v_h\|_{L^2(\Omega)}^2 \leq B(\omega_h, v_h; \tau_h, w_h).$$

This shows (A.7). We clearly have (A.8) if we use (A.10). \square

Using the above stability result we can prove an a-priori error estimate.

Theorem A.6. *Let σ, u solve (2.6) and $\tilde{\sigma}_h^R, \tilde{u}_h^R$ solve (A.6), then*

$$\begin{aligned} & \|\tilde{\Pi}^R \sigma - \tilde{\sigma}_h^R\|_{\mathcal{A}} + \|\tilde{P}^R \operatorname{div}(\tilde{\Pi}^R \sigma - \tilde{\sigma}_h^R)\|_{L^2(\Omega)}^2 + \|\tilde{P}^R u - \tilde{u}_h^R\|_{L^2(\Omega)} \\ & \leq C(\|\tilde{\Pi}^R \sigma - \sigma\|_{\mathcal{A}} + \|\tilde{P}^R u - u\|_{L^2(\Omega)}). \end{aligned}$$

Proof. By Theorem A.5, there exist $\tau_h \in \tilde{\Sigma}_h^R$ and $w_h \in \tilde{V}_h^R$, satisfying

$$\begin{aligned} & \|\tilde{\Pi}^R \sigma - \tilde{\sigma}_h^R\|_{\mathcal{A}}^2 + \|\tilde{P}^R \operatorname{div}(\tilde{\Pi}^R \sigma - \tilde{\sigma}_h^R)\|_{L^2(\Omega)}^2 + \|\tilde{P}^R u - \tilde{u}_h^R\|_{L^2(\Omega)}^2 \\ & \leq c_1 B(\tilde{\Pi}^R \sigma - \tilde{\sigma}_h^R, \tilde{P}^R u - \tilde{u}_h^R; \tau_h, w_h) \end{aligned}$$

and

$$\begin{aligned} & \|\tau_h\|_{\mathcal{A}}^2 + \|\tilde{P}^R \operatorname{div} \tau_h\|_{L^2(\Omega)}^2 + \|w_h\|_{L^2(\Omega)}^2 \\ \text{(A.11)} \quad & \leq c_2(\|\tilde{\Pi}^R \sigma - \tilde{\sigma}_h^R\|_{\mathcal{A}}^2 + \|\tilde{P}^R \operatorname{div}(\tilde{\Pi}^R \sigma - \tilde{\sigma}_h^R)\|_{L^2(\Omega)}^2 + \|\tilde{P}^R u - \tilde{u}_h^R\|_{L^2(\Omega)}^2). \end{aligned}$$

Then,

$$\begin{aligned} & \|\tilde{\Pi}^R \sigma - \tilde{\sigma}_h^R\|_{\mathcal{A}}^2 + \|\tilde{P}^R \operatorname{div}(\tilde{\Pi}^R \sigma - \tilde{\sigma}_h^R)\|_{L^2(\Omega)}^2 + \|\tilde{P}^R u - \tilde{u}_h^R\|_{L^2(\Omega)}^2 \\ & \leq c_1 \left((\mathcal{A}(\tilde{\Pi}^R \sigma - \tilde{\sigma}_h^R, \tau_h) + (\tilde{P}^R u - \tilde{u}_h^R, \operatorname{div} \tau_h)) \right) \end{aligned}$$

where we used (2.6b), (A.6b) and (A.3) to say the term $(\operatorname{div}(\tilde{\Pi}^R \sigma - \tilde{\sigma}_h^R), w_h)$ vanishes.

On the other hand, using (2.6a), (A.6a) we get

$$(\mathcal{A}(\tilde{\Pi}^R \sigma - \tilde{\sigma}_h^R), \tau_h) + (\tilde{P}^R u - \tilde{u}_h^R, \operatorname{div} \tau_h) = (\mathcal{A}(\tilde{\Pi}^R \sigma - \sigma), \tau_h) + (\tilde{P}^R u - u, \operatorname{div} \tau_h).$$

The result now follows by using (A.11). \square

As a corollary, we give an error estimate with convergence rate.

Corollary A.7. *Let σ, u solve (2.6) and $\tilde{\sigma}_h^R, \tilde{u}_h^R$ solve (A.6). If $\sigma \in H^1(\Omega, \mathbb{S}), u \in [H^1(\Omega)]^3$, then*

$$\text{(A.12)} \quad \|\sigma - \tilde{\sigma}_h^R\|_{\mathcal{A}} + \|u - \tilde{u}_h^R\|_{L^2(\Omega)} \leq Ch(\|\sigma\|_{H^1(\Omega)} + \|u\|_{H^1(\Omega)}).$$

Remark A.8. The argument in the proof of Theorem 2.11 (on robustness near incompressibility) does not extend to this reduced space pair because an analogue of (4.8) is not available for $\tilde{\sigma}_h^R$ and $\tilde{\Pi}^R$.

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PORTLAND STATE UNIVERSITY, PO BOX 751 (MTH), PORTLAND, OR 97207, USA
Email address: gjay@pdx.edu

DIVISION OF APPLIED MATHEMATICS, BROWN UNIVERSITY, BOX F, 182 GEORGE STREET,
 PROVIDENCE, RI 02912, USA
Email address: johnny_guzman@brown.edu

DEPARTMENT OF MATHEMATICS, BAYLOR UNIVERSITY, SID RICHARDSON SCIENCE BUILDING,
 1410 S.4TH STREET, WACO, TX 76706, USA
Email address: jeonghun.lee@baylor.edu