A 2-COMPLEX CONTAINING SOBOLEV SPACES OF MATRIX FIELDS

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Abstract. Using a generalization of complexes, called 2-complexes, this paper defines and analyzes new Sobolev spaces of matrix fields and their interrelationships within a commuting diagram. These spaces have very weak second-order derivatives. An example is the space of matrix fields of square-integrable components whose row-wise divergence followed by yet another divergence operation yield a function in a standard negative-order Sobolev space. Similar spaces where the double divergence is replaced by a curl composed with divergence, or a double curl operator (the incompatibility operator), are also studied. Stable decompositions of such spaces in terms of more regular component functions (which are continuous in natural norms) are established. Appropriately ordering such Sobolev spaces with and without boundary conditions (in a weak sense), we discover duality relationships between them. Motivation to study such Sobolev spaces, from a finite element perspective and implications for weak well-posed variational formulations are pointed out.

Keywords: Sobolev spaces, regular decomposition, 2-complex, Hilbert complexes.

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1. Introduction

Substantial improvements in numerical techniques for solving partial differential equations (PDEs) to address current scientific challenges have come from connections to and preservation of the differential and algebraic structures inherent in the PDEs. Ample examples are offered by the history of finite element techniques. The earliest finite elements [17], Lagrange elements, consisted of *scalar-valued* functions. Developments in *vector-valued* finite elements followed, starting with elements [39] of continuous normal (n) components. These "n-continuous" elements were supplemented with "t-continuous" vector-valued Nédélec elements with continuous tangential (t) components [32]. Further families of vector-valued elements were unearthed continuing this line of work. Although these elements were developed separately, today we understand them together as fitting into a cochain subcomplex of a de Rham complex of Sobolev spaces, thanks to intensive research into finite element exterior calculus (FEEC) [4, 6, 25]. It is now clear how to generalize from scalar and vector fields to tensor fields, as long as the tensors have the algebraic structure of k-forms in the de Rham complex, i.e., higher order *alternating tensor-valued* finite elements in any dimension naturally fit into FEEC.

This paper, while building on these developments, is motivated by other types of tensors. Problems in continuum mechanics, differential geometry and general relativity call for a study of tensors with other types of symmetries. Indeed, even restricting to second-order tensors, the need for study is evident from the increasing current interest in matrix-valued finite element functions. The earliest of these are the "nn-continuous" symmetric matrix fields (i.e., symmetric matrix-valued functions σ with continuous $(\sigma n) \cdot n$) of the HHJ (Hellan-Herrmann-Johnson) elements [15], now enjoying a revival [7,37,38,40] in the TDNNS method and elsewhere. A seemingly disjoint (but potentially connected) recent development is the "nt-continuous" trace-free matrix finite element developed [19–21] for viscous fluid stresses in the context of the MCS (Mass-Conserving Stress-yielding) method. To add to this picture, Regge elements [14, 22, 23, 29] with "tt-continuous" symmetric matrix-valued elements are finding more and more uses. How does one connect these disparate developments of nn, nt, and tt-continuous matrix finite elements? The prior synthesis (mentioned in the previous paragraph) involved spaces of the de Rham complex, all connected by fundamental first-order differential operators (grad, curl, and divergence, in three dimensions). In contrast, what seems to be natural for the matrix finite elements are other *second-order* differential operators.

The goal of this work is to take a step toward understanding what *Sobolev spaces* and their arrangements might reveal a unified structure where such second-order differential operators and matrix fields arise naturally. Although motivated by finite elements, this work does not contain finite elements. The scope is limited to a study of infinite-dimensional Sobolev spaces of matrix fields, their interrelationships, and connections to standard Sobolev spaces. We focus on spaces of scalar, vector, and matrix valued functions on *three-dimensional* (3D) domains Ω. Study of higher order tensor fields on higher dimensional domains is certainly interesting, but requires more algebraic machinery (such as group representations and Young tableaux) to work with tensor symmetries.

In 3D however, the relevant symmetries can be captured by the familiar symmetrization and deviatoric operations,

$$\operatorname{sym} \tau = \frac{1}{2}(\tau + \tau^\top), \qquad \operatorname{dev} \tau = \tau - \frac{1}{3}\operatorname{tr}(\tau)\mathbb{I}, \qquad \tau \in \mathbb{M}, \tag{1}$$

where $\mathbb{M} = \mathbb{R}^{3\times 3}$ denotes the vector space of 3×3 real ($\mathbb{R}$) matrices, $\operatorname{tr}(\tau)$ denotes the trace of a matrix $\tau \in \mathbb{M}$, and $\mathbb{I}$ denotes the 3×3 identity matrix. Here and throughout, $\tau^\top$, also written as $\top\tau$, denotes the (pointwise) transpose of a matrix field τ. The operations in (1) generate subspaces of symmetric matrices and trace-free matrices which we denote by

$$\mathbb{S} = \operatorname{sym} \mathbb{M}, \qquad \mathbb{T} = \operatorname{dev} \mathbb{M}.$$

Let $\mathbb{V} = \mathbb{R}^3$. We are interested in structures connecting Sobolev spaces of functions with values in $\mathbb{R}$, $\mathbb{V}$, $\mathbb{S}$ and $\mathbb{T}$ of the following form:

$$\begin{array}{ccccccc}
\mathbb{R} & \longrightarrow & \mathbb{V} & \longrightarrow & \mathbb{V} & \longrightarrow & \mathbb{R} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{V} & \longrightarrow & \mathbb{S} & \longrightarrow & \mathbb{T} & \longrightarrow & \mathbb{V} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{V} & \longrightarrow & \mathbb{T} & \longrightarrow & \mathbb{S} & \longrightarrow & \mathbb{V} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{R} & \longrightarrow & \mathbb{V} & \longrightarrow & \mathbb{V} & \longrightarrow & \mathbb{R}
\end{array} \tag{2}$$

Such diagrams where $\mathbb{R}$, $\mathbb{V}$, $\mathbb{S}$ and $\mathbb{T}$ are replaced by appropriate Sobolev spaces of functions taking values in them, are studied here. The first such diagram is introduced below in (25), which contain first-order derivative operators as well as key algebraic operations $\top$, sym, and dev. Certain combinations of these operations result in basic second-order derivative operators marked in diagram (37).

The tensors along the four edges of (2) follow the $\mathbb{R}$-$\mathbb{V}$-$\mathbb{V}$-$\mathbb{R}$ pattern of the well-known 3D de Rham complex

$$C^\infty \xrightarrow{\text{grad}} C^\infty \otimes \mathbb{V} \xrightarrow{\text{curl}} C^\infty \otimes \mathbb{V} \xrightarrow{\text{div}} C^\infty \tag{3}$$

of infinitely smooth (C^∞) scalar and vector fields on Ω. Recall that a "complex" is a sequence of linear spaces X_i and linear maps $A_i : X_i \to X_{i+1}$, traditionally expressed by

$$\cdots \quad X_{k-2} \xrightarrow{A_{k-2}} X_{k-1} \xrightarrow{A_{k-1}} X_k \xrightarrow{A_k} X_{k+1} \xrightarrow{A_{k+1}} X_{k+2} \quad \cdots, \tag{4}$$

satisfying $A_{i+1} \circ A_i = 0$ for all i. In [33], "ℓ-complexes" arose, which are sequences (4) with the property $A_{i+\ell} \circ \cdots \circ A_{i+1} \circ A_i = 0$ for all i and some fixed integer ℓ (so, e.g., a 1-complex is a complex in the usual sense). As we shall see, diagrams of the form (2) that we study here have a 2-complex structure (which explains the title of this paper). Definition 1.1 below formalizes the 2-complex notion in the context of such diagrams.

Other examples of complexes, beyond the de Rham complex (3), include the well-known elasticity complex [2, 5, 8, 28, 35], also named after Calabi or Kröner,

$$C^\infty \otimes \mathbb{V} \xrightarrow{\text{sym grad}} C^\infty \otimes \mathbb{S} \xrightarrow{\text{curl} \top \text{curl}} C^\infty \otimes \mathbb{S} \xrightarrow{\text{div}} C^\infty \otimes \mathbb{V}, \tag{5}$$

the hessian complex [2, 26]

$$C^\infty \xrightarrow{\text{grad grad}} C^\infty \otimes \mathbb{S} \xrightarrow{\text{curl}} C^\infty \otimes \mathbb{T} \xrightarrow{\text{div}} C^\infty \otimes \mathbb{V}, \tag{6}$$

and the div div complex [2, 34]

$$C^\infty \otimes \mathbb{V} \xrightarrow{\text{dev grad}} C^\infty \otimes \mathbb{T} \xrightarrow{\text{sym curl}} C^\infty \otimes \mathbb{S} \xrightarrow{\text{div div}} C^\infty. \tag{7}$$

These complexes can be systematically derived from the de Rham complex (3) using the Bernstein-Gelfand-Gelfand (BGG) construction, originally developed in algebraic and geometric contexts [9, 13] and more recently adapted to certain Sobolev spaces [2, 12]. We shall see that analogous complexes, with other "H^{-1} based" Sobolev spaces, defined shortly in (23)–(24), also arise naturally from the 2-complexes and the diagrams of the type (2) that we study here.

On the theme of *weakly regular H^{-1} based Sobolev spaces,* which is pervasive in this paper, some motivating examples shed more light. Let

$$(u, v) := \int_\Omega uv \, dx \tag{8}$$

for scalar fields u, v, and in addition, for vector or matrix fields u and v, we continue to use the same notation (u, v) to denote the Lebesgue integral (when it exists) over Ω of the dot product $u \cdot v$, or the Frobenius product $u : v$, respectively, of u and v. All function spaces on Ω are defined precisely in Subsection 1.1, but for expediency, we use the standard space L_2 and the space H^{-1} with weaker topology in the quick discussion of two examples below, both showing the role of weak regularity, and each illuminating the role of one of two algebraic operations sym and dev.

Example 1. The stress tensor σ in linear elasticity is a matrix field which must satisfy $\sigma = \operatorname{sym}(\sigma)$ due to conservation of angular momentum. Well-posed formulations for the Hellinger-Reissner principle in linear elasticity seek a symmetric matrix field (the stress tensor) $\sigma : \Omega \to \mathbb{S}$ in some Sobolev space Σ and a vector field $u : \Omega \to \mathbb{V}$ (displacement) in some Sobolev space V satisfying

$$\begin{aligned} (A\sigma, \tau) + (u, \operatorname{div}\tau) &= 0 \qquad \text{for all } \tau \in \Sigma, \\ (\operatorname{div}\sigma, v) &= (f, v) \quad \text{for all } v \in V, \end{aligned} \tag{9}$$

where A, f, and div denotes the compliance tensor, the load vector field, and row-wise divergence of a matrix field, respectively. A "regular choice" is $V = L_2 \otimes \mathbb{V}$ and

$$\Sigma = \{\tau \in L_2 \otimes \mathbb{S} : \operatorname{div}\tau \in L_2 \otimes \mathbb{V}\}. \tag{10}$$

Construction of finite elements for this Σ is difficult and had remained an open problem for decades, as noted in [3]. (If the symmetry condition on σ were absent, then three copies of the n-continuous finite elements would have been sufficient.) An alternative choice of "weak regularity" is

$$\Sigma = \{\tau \in L_2 \otimes \mathbb{S} : \operatorname{div}\tau \in V^*\} \tag{11}$$

where V^* is weaker than L_2 integrals of the form $(\operatorname{div}\tau, v)$ in (9) are relaxed to a duality pairing $(\operatorname{div}\tau)(v)$ in V. The TDNNS method [37,38] with nn-continuous stresses can be seen as a discretization of such a formulation with $V = \mathring{H}(\text{curl})$, a space defined shortly in (15). Theorem 5.4 shows that the condition $\operatorname{div}\tau \in V^*$ in (11) is equivalent to $\operatorname{div}\operatorname{div}\tau \in H^{-1}$. This motivates us to study Sobolev spaces with this weak regularity condition, namely the spaces H_{dd} and $\mathcal{H}_{\text{dd}}$ defined in (73c) and (74b), respectively.

Example 2. Viscous stresses in Stokes flow with fluid velocity $u : \Omega \to \mathbb{V}$ can be extracted from the symmetric part of $\sigma = 2\nu \operatorname{grad} u$, where ν is the kinematic viscosity. The incompressibility constraint $\operatorname{div} u = 0$, a well known source of challenges in numerical simulation [27], now emerges as an algebraic constraint: $\sigma = \operatorname{dev}\sigma$. The definition of σ and flow equations suggest that we should find σ in a space Σ of trace-free matrix fields, u in some space V of vector fields, and the pressure p in some space Q of scalar fields such that

$$\begin{aligned} (\nu^{-1}\sigma, \tau) + (u, \operatorname{div}\tau) &= 0 && \text{for all } \tau \in \Sigma, \\ (\operatorname{div}\sigma, v) + (\operatorname{div} v, p) &= -(f, v) && \text{for all } v \in V, \\ (\operatorname{div} u, q) &= 0 && \text{for all } p \in Q. \end{aligned}$$

for some given source field f. The MCS method [19–21] sets $V = \mathring{H}(\text{div})$ (a space defined shortly in (15)) and $Q = \operatorname{div} V$. Then, instead of a "regular choice" $\Sigma = \{\tau \in L_2 \otimes \mathbb{T} : \operatorname{div}\sigma \in L_2 \otimes \mathbb{V}\}$ that would make the integrals like $(\operatorname{div}\sigma, v)$ well defined, the MCS formulation proposes a choice of "weak regularity," namely

$$\Sigma = \{\tau \in L_2 \otimes \mathbb{T} : \operatorname{div}\sigma \in V^*\} \tag{12}$$

for which simple nt-continuous finite elements work, after relaxing $(\operatorname{div}\sigma, v)$ to a duality pairing $(\operatorname{div}\sigma)(v)$ in V. Theorem 5.4 shows that the condition $\operatorname{div}\tau \in V^*$ in (12) is equivalent to $\operatorname{curl}\operatorname{div}\tau \in H^{-1}$. This motivates us to study Sobolev spaces with this weak regularity condition, namely the spaces H_{cd} and $\mathcal{H}_{\text{cd}}$ defined in (73b) and (74c), respectively.

1.1. **Preliminaries and spaces.** Let Ω be a bounded open connected subset of the Euclidean space $\mathbb{R}^3$ with Lipschitz boundary. Let L_2 denote the space of square-integrable $\mathbb{R}$-valued functions on Ω, or equivalently the space of square-integrable $\mathbb{R}$-valued functions on $\mathbb{R}^3$ supported on $\bar{\Omega}$. Let $\mathcal{D}(\Omega)$ denote the Schwartz space of smooth test functions on Ω that are compactly supported in Ω. The dual of any topological space X is denoted by X^*. The space of distributions on Ω is denoted by $\mathcal{D}(\Omega)^*$. The space of vector fields on Ω with square-integrable components is denoted by $L_2 \otimes \mathbb{V}$ and the notation is similarly extended to $\mathbb{T}$ and $\mathbb{S}$-valued fields on Ω as well as to other spaces, e.g., $\mathcal{D}(\Omega)^* \otimes \mathbb{S}$ denotes the space of symmetric matrix-valued fields whose components are distributions on Ω.

We use the standard Sobolev spaces $H^s(\mathbb{R}^3)$ and $H^s(\Omega)$ for any $s \in \mathbb{R}$. (see e.g., [1, 30]). We omit the domain Ω from the notation when no confusion can arise and simply write H^s for $H^s(\Omega)$. For scalar functions $u : \Omega \to \mathbb{R}$ and vector functions $v, q : \Omega \to \mathbb{V}$, let

$$\begin{gathered} \|u\|_{H^1}^2 = \|u\|_{L_2}^2 + \|\operatorname{grad} u\|_{L_2}^2, \\ \|v\|_{H(\operatorname{curl})}^2 = \|v\|_{L_2}^2 + \|\operatorname{curl} v\|_{L_2}^2, \qquad \|q\|_{H(\operatorname{div})}^2 = \|q\|_{L_2}^2 + \|\operatorname{div} v\|_{L_2}^2. \end{gathered} \tag{13}$$

These are norms of well-known Hilbert spaces of functions on Ω, namely

$$H(\operatorname{grad}) \equiv H^1 = \{u \in L_2 : \operatorname{grad} u \in L_2 \otimes \mathbb{V}\}, \tag{14a}$$

$$H(\operatorname{curl}) = \{v \in L_2 \otimes \mathbb{V} : \operatorname{curl} v \in L_2 \otimes \mathbb{V}\}, \tag{14b}$$

$$H(\operatorname{div}) = \{q \in L_2 \otimes \mathbb{V} : \operatorname{div} q \in L_2\}. \tag{14c}$$

Using the standard norms in (13), the closures

$$\mathring{H}(\operatorname{grad}) = \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{H^1}}, \quad \mathring{H}(\operatorname{curl}) = \overline{\mathcal{D}(\Omega) \otimes \mathbb{V}}^{\|\cdot\|_{H(\operatorname{curl})}}, \quad \mathring{H}(\operatorname{div}) = \overline{\mathcal{D}(\Omega) \otimes \mathbb{V}}^{\|\cdot\|_{H(\operatorname{div})}}, \tag{15}$$

give well-known zero-trace subspaces of the spaces in (14).

Further spaces are defined using similar closures, but using the set of $\mathcal{D}(\Omega)$-functions extended by zero to all $\mathbb{R}^3$ and closing the set using the $H^s(\mathbb{R}^3)$ norm. Set

$$\mathring{\widetilde{H}}^s = \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{H^s(\mathbb{R}^3)}}, \qquad s \in \mathbb{R}, \tag{16}$$

and $\|u\|_{\mathring{\widetilde{H}}^s} := \|u\|_{H^s(\mathbb{R}^3)}$. This space is often just denoted by $\widetilde{H}^s(\Omega)$, and in our setting, is also the same as another often-occurring space in the literature, $H^s_{\overline{\Omega}}(\mathbb{R}^3) = \{u \in H^s(\mathbb{R}^3) : \operatorname{supp} u \subset \bar{\Omega}\}$ (see e.g. [30, Theorem 3.29]), i.e.,

$$\mathring{\widetilde{H}}^s = \{u \in H^s(\mathbb{R}^3) : \ \operatorname{supp} u \subset \bar{\Omega}\}. \tag{17}$$

Since the ith partial derivative ∂_i satisfies $\|\partial_i \varphi\|_{\mathring{\widetilde{H}}^s} \le \|\varphi\|_{\mathring{\widetilde{H}}^{s+1}}$ for all $\varphi \in \mathcal{D}(\Omega)$, and since $\mathcal{D}(\Omega)$ is dense in $\mathring{\widetilde{H}}^s$ by definition (16), we conclude that for any real s,

$$\partial_i : \mathring{\widetilde{H}}^{s+1} \to \mathring{\widetilde{H}}^s \quad \text{is continuous.} \tag{18}$$

It is well known [30, Theorem 3.30] that $\mathring{\widetilde{H}}^s$ is also identifiable with a standard dual space

$$\mathring{\widetilde{H}}^s = (H^{-s})^* \tag{19}$$

for any $s \in \mathbb{R}$. The case $s = -1$ is of particular interest here. The space $\mathring{\widetilde{H}}^{-1}$, not to be confused with $H^{-1} = \mathring{H}(\operatorname{grad})^*$, satisfies, per (19),

$$\mathring{\widetilde{H}}^{-1} = H(\operatorname{grad})^*, \tag{20}$$

and furthermore, even if $\mathring{\widetilde{H}}^{-1}$ is not embedded in a space of distributions on Ω, it can be characterized using tempered distributions on $\mathbb{R}^3$ supported on the closure of Ω, due to (17).

Therefore the norm of any u in $\mathring{\widetilde{H}}^{-1}$ can be computed either using the $H^{-1}(\mathbb{R}^3)$-norm of the extension of u by zero to all $\mathbb{R}^3$, or by duality using (20). Finally, we note that it is also well-known [30, Theorem 3.33] when $s > 0$, $\mathring{\widetilde{H}}^s$ is contained in

$$\mathring{H}^s := \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{H^s(\Omega)}}$$

and, moreover, $\mathring{\widetilde{H}}^s$ and $\mathring{H}^s$ are equal if $s > 0$ and $s - \frac{1}{2}$ is not an integer, so e.g., $\mathring{H}^1 = \mathring{\widetilde{H}}^1$.

For a general $s \in \mathbb{R}$, we define the norms

$$\|v\|^2_{\mathring{\widetilde{H}}^s(\mathrm{curl})} = \|v\|^2_{\mathring{\widetilde{H}}^s} + \|\operatorname{curl} v\|^2_{\mathring{\widetilde{H}}^s}, \qquad \|q\|^2_{\mathring{\widetilde{H}}^s(\mathrm{div})} = \|q\|^2_{\mathring{\widetilde{H}}^s} + \|\operatorname{div} q\|^2_{\mathring{\widetilde{H}}^s}.$$

and set

$$\mathring{\widetilde{H}}^s(\mathrm{curl}) = \overline{\mathcal{D}(\Omega) \otimes \mathbb{V}}^{\|\cdot\|_{\mathring{\widetilde{H}}^s(\mathrm{curl})}}, \qquad \mathring{\widetilde{H}}^s(\mathrm{div}) = \overline{\mathcal{D}(\Omega) \otimes \mathbb{V}}^{\|\cdot\|_{\mathring{\widetilde{H}}^s(\mathrm{div})}}. \tag{21}$$

These spaces with $s = -1$ feature in a central diagram introduced shortly.

Next we introduce key spaces of matrix-valued fields, which are also needed for the diagram. Note that when the standard differential operators div and curl are applied to matrix-valued fields, we do so row-wise. The next definitions involve second-order differential operators on matrix-valued functions $g : \Omega \to \mathbb{S}$, $\tau : \Omega \to \mathbb{T}$, and $\sigma : \Omega \to \mathbb{S}$, such as the incompatibility operator

$$\operatorname{inc} g := \operatorname{curl} \top \operatorname{curl} g. \tag{22}$$

Let

$$\|g\|^2_{\mathring{\widetilde{H}}_{\mathrm{cc}}} = \|g\|^2_{\mathring{\widetilde{H}}^{-1}} + \|\operatorname{curl} g\|^2_{\mathring{\widetilde{H}}^{-1}} + \|\operatorname{inc} g\|^2_{\mathring{\widetilde{H}}^{-1}} \tag{23a}$$

$$\|\tau\|^2_{\mathring{\widetilde{H}}_{\mathrm{cd}}} = \|\tau\|^2_{\mathring{\widetilde{H}}^{-1}} + \|\operatorname{div} \tau\|^2_{\mathring{\widetilde{H}}^{-1}} + \|\operatorname{sym}\operatorname{curl} \top \tau\|^2_{\mathring{\widetilde{H}}^{-1}} + \|\operatorname{curl}\operatorname{div} \tau\|^2_{\mathring{\widetilde{H}}^{-1}} \tag{23b}$$

$$\|\sigma\|^2_{\mathring{\widetilde{H}}_{\mathrm{dd}}} = \|\sigma\|^2_{\mathring{\widetilde{H}}^{-1}} + \|\operatorname{div} \sigma\|^2_{\mathring{\widetilde{H}}^{-1}} + \|\operatorname{div}\operatorname{div} \sigma\|^2_{\mathring{\widetilde{H}}^{-1}}, \tag{23c}$$

and $\|\tau\|_{\mathring{\widetilde{H}}_{\mathrm{cd}\top}} = \|\top \tau\|_{\mathring{\widetilde{H}}_{\mathrm{cd}}}$. Let

$$\mathring{\widetilde{H}}_{\mathrm{cc}} = \overline{\mathcal{D}(\Omega) \otimes \mathbb{S}}^{\|\cdot\|_{\mathring{\widetilde{H}}_{\mathrm{cc}}}}, \quad \mathring{\widetilde{H}}_{\mathrm{cd}} = \overline{\mathcal{D}(\Omega) \otimes \mathbb{T}}^{\|\cdot\|_{\mathring{\widetilde{H}}_{\mathrm{cd}}}}, \quad \mathring{\widetilde{H}}_{\mathrm{dd}} = \overline{\mathcal{D}(\Omega) \otimes \mathbb{S}}^{\|\cdot\|_{\mathring{\widetilde{H}}_{\mathrm{dd}}}}. \tag{24}$$

The space $\mathring{\widetilde{H}}_{\mathrm{cd}\top} = \{\tau^\top : \tau \in \mathring{\widetilde{H}}_{\mathrm{cd}}\}$ will also be needed. Clearly, in view of (16), the spaces $\mathring{\widetilde{H}}_{\mathrm{cc}}, \mathring{\widetilde{H}}_{\mathrm{cd}}$ and $\mathring{\widetilde{H}}_{\mathrm{dd}}$ are subspaces of $\mathring{\widetilde{H}}^{-1} \otimes \mathbb{S}$, $\mathring{\widetilde{H}}^{-1} \otimes \mathbb{T}$, and $\mathring{\widetilde{H}}^{-1} \otimes \mathbb{S}$, respectively.

Certain subspaces of $\mathring{\widetilde{H}}^s$, $\mathring{\widetilde{H}}^s(\mathrm{curl})$ and $\mathring{\widetilde{H}}^s(\mathrm{div})$, which we now define, occur often. Let $\mathcal{P}_1$ denote the space of linear polynomials. Using the coordinate vector x in $\mathbb{R}^3$, define

$$\mathcal{RT} = \{a + bx : a \in \mathbb{V}, b \in \mathbb{R}\}, \qquad \mathcal{ND} = \{a + d \times x : a, d \in \mathbb{V}\}.$$

Let

$$L_{2,\mathbb{R}} = \{u \in L_2 : (u, 1) = 0\}, \quad \mathring{\widetilde{H}}^s_{\mathbb{R}} = \{u \in \mathring{\widetilde{H}}^s : u(1) = 0\},$$

$$\mathring{\widetilde{H}}^s_{\mathcal{RT}}(\mathrm{curl}) = \{v \in \mathring{\widetilde{H}}^s(\mathrm{curl}) : v(r) = 0 \text{ for all } r \in \mathcal{RT}\},$$

$$\mathring{\widetilde{H}}^s_{\mathcal{ND}}(\mathrm{div}) = \{q \in \mathring{\widetilde{H}}^s(\mathrm{div}) : q(r) = 0 \text{ for all } r \in \mathcal{ND}\},$$

$$\mathring{\widetilde{H}}^s_{\mathcal{P}_1} = \{w \in \mathring{\widetilde{H}}^s : w(p) = 0 \text{ for all } p \in \mathcal{P}_1\}.$$

Here and throughout, the action of a distribution w on a function p in $\mathcal{D}(\mathbb{R}^3)$ is denoted by $w(p)$. In the above subspaces of distributions, note that only the value of $p|_\Omega$ on Ω is needed to evaluate the action $w(p)$ since w is supported on $\bar{\Omega}$. Note also that $\mathring{\widetilde{H}}^s_{\mathcal{RT}}(\mathrm{curl})$ and $\mathring{\widetilde{H}}^s_{\mathcal{ND}}(\mathrm{div})$ are closed subspaces of $\mathring{\widetilde{H}}^s(\mathrm{curl})$ and $\mathring{\widetilde{H}}^s(\mathrm{div})$.

1.2. **A diagram connecting the Sobolev spaces.** Using the above-defined notation, we can now precisely introduce one of the objects of study in this paper. It is the following diagram connecting the above-defined Sobolev spaces of scalar-, vector-, and matrix-valued distributions on $\mathbb{R}^3$:

$$
\begin{array}{ccccccc}
\mathring{H}(\mathrm{grad}) & \xrightarrow{\ \mathrm{grad}\ } & \mathring{H}(\mathrm{curl}) & \xrightarrow{\ \mathrm{curl}\ } & \mathring{H}(\mathrm{div}) & \xrightarrow{\ \mathrm{div}\ } & L_{2,\mathbb{R}} \\
\Big\downarrow{\scriptstyle \mathrm{grad}} & & \Big\downarrow{\scriptstyle \mathrm{def}} & & \Big\downarrow{\scriptstyle \frac{1}{2}\top\,\mathrm{dev\,grad}} & & \Big\downarrow{\scriptstyle \frac{1}{3}\mathrm{grad}} \\
\mathring{H}(\mathrm{curl}) & \xrightarrow{\ \mathrm{def}\ } & \mathring{\tilde{H}}_{\mathrm{cc}} & \xrightarrow{\ \mathrm{curl}\ } & \mathring{\tilde{H}}_{\mathrm{cd}} & \xrightarrow{\ \mathrm{div}\ } & \mathring{\tilde{H}}^{-1}_{\mathcal{RT}}(\mathrm{curl}) \\
\Big\downarrow{\scriptstyle \mathrm{curl}} & & \Big\downarrow{\scriptstyle \top\,\mathrm{curl}} & & \Big\downarrow{\scriptstyle \mathrm{sym\,curl}\,\top} & & \Big\downarrow{\scriptstyle \frac{1}{2}\mathrm{curl}} \\
\mathring{H}(\mathrm{div}) & \xrightarrow{\ \frac{1}{2}\mathrm{dev\,grad}\ } & \mathring{\tilde{H}}_{\mathrm{cd}\top} & \xrightarrow{\ \mathrm{sym\,curl}\ } & \mathring{\tilde{H}}_{\mathrm{dd}} & \xrightarrow{\ \mathrm{div}\ } & \mathring{\tilde{H}}^{-1}_{\mathcal{ND}}(\mathrm{div}) \\
\Big\downarrow{\scriptstyle \mathrm{div}} & & \Big\downarrow{\scriptstyle \mathrm{div}\,\top} & & \Big\downarrow{\scriptstyle \mathrm{div}} & & \Big\downarrow{\scriptstyle \mathrm{div}} \\
L_{2,\mathbb{R}} & \xrightarrow{\ \frac{1}{3}\mathrm{grad}\ } & \mathring{\tilde{H}}^{-1}_{\mathcal{RT}}(\mathrm{curl}) & \xrightarrow{\ \frac{1}{2}\mathrm{curl}\ } & \mathring{\tilde{H}}^{-1}_{\mathcal{ND}}(\mathrm{div}) & \xrightarrow{\ \mathrm{div}\ } & \mathring{\tilde{H}}^{-1}_{\mathcal{P}_1}
\end{array}
\tag{25}
$$

Here $\mathrm{def}\, u = \mathrm{sym}\,\mathrm{grad}\, u$ for vector fields u denotes the deformation operator, where $\mathrm{grad}\, u$ is the matrix field whose (i, j)th component is $\partial u_i/\partial x_j$. Note that information in the diagram (25) is repeated across the diagonal, i.e., the diagram is symmetric about the diagonal. The properties collected in the next section show that each of the indicated operators is linear and continuous in the norms of the indicated domain and codomain, and that each component cell in the diagram commutes. A different but similar diagram starting with analogous *spaces without boundary conditions* $H(\mathrm{grad}), H(\mathrm{curl})$, and $H(\mathrm{div})$, is found later in Section 5.

In the commutative diagram (25), the "objects" (or "vertices") are the spaces. The "morphisms" (or "arrows") are the indicated first-order differential operators. Compositions of morphisms are referred to as "paths". Clearly, paths in (25) always go right or down from an object. The following definition of a "2-complex" is motivated by [33].

Definition 1.1. A path is a **complex** if the composition of two successive morphisms in it vanish. We say that a path is a **2-complex** if the composition of three successive morphisms in it vanish.

We show (in the next section, in Theorem 2.4) that all paths in the diagram (25) are 2-complexes. The analogous diagram for spaces without boundary conditions also shares the same property, as we shall see in Section 5.

Before concluding this introduction, a few remarks on comparison with the BGG approach are in order. The BGG construction of [2, 12] produces analogues of (5), (6) and (7) with Sobolev spaces $H^q \otimes \mathbb{W}$ or $H(D, \mathbb{W}) := \{\sigma \in L_2 \otimes \mathbb{W} : D\sigma \in L_2 \otimes \tilde{\mathbb{W}}\}$, for appropriate $\mathbb{W}, \tilde{\mathbb{W}} \in \{\mathbb{S}, \mathbb{T}\}$ and operators D from the above complexes. The hessian, elasticity, and div-div complexes were also studied individually in other works [18, 34, 36]. It should not be surprising that some individual results in this paper may be alternately derived using the prior approaches, e.g., the commutativity identities (26) are extensively used in BGG works, and the regular decomposition for two of the "slightly more regular spaces" in Section 4, $\hat{\tilde{\mathcal{H}}}_{\mathrm{cc}}$ and $\hat{\tilde{\mathcal{H}}}_{\mathrm{dd}}$, can be approached using the technique of [2, Theorem 3] with minor changes. However, such individual results do not fully address the objectives of this paper. For instance, the spaces defined in (23) do not emerge from [2, 12] as canonical spaces with a unified definition; rather, they exhibit a cohesive pattern only through the perspective of the 2-complexes in (25). Consequently, the analytical results for these spaces, such as regular decompositions, differ significantly from those in [2, 12]. Moreover, the 2-complex in (25) unifies several key spaces, including the Hessian, elasticity, and div div complexes, potentially inspiring novel constructions across diverse applications. This unification can be reminiscent of the BGG diagram [2, 12]. However, a critical distinction is that BGG diagrams involve full matrix spaces requiring subsequent symmetry reduction, whereas (25) directly incorporates spaces of tensors with the symmetrizations.

1.3. **Outline.** The next section (Section 2) begins by gathering a number of identities from which the commutativity properties in the diagram (25) become evident. We prove the 2-complex property of (25), show how the elasticity complex, the hessian complex and the div-div complex emerges from the diagram. In Section 3, we prove that the newly introduced H^{-1} based Sobolev spaces of weak regularity admit decompositions with smoother component functions that vary continuously with the decomposed function (Theorems 3.4, 3.7, and 3.10). We construct right inverses (in Theorem 3.21) of the operators in (25) as well as of second-order differential operators that emerge from the diagram, from which it follows that the ranges of the differential operators considered are closed. This can be used to prove exactness of derived complexes. Slightly smoother versions of the matrix-valued Sobolev spaces are then considered in Section 4 and shorter regular decompositions for them are proved. Finally, in Section 5, we mention extensions to the case of analogous spaces without boundary conditions. The main result of that section is Theorem 5.4 which shows how the diagrams of spaces with and without boundary conditions are in correspondence through duality.

2. Continuity, commutativity, and 2-complex properties

In this section, we show that the diagram (25) is a commuting diagram and has the 2-complex property.

In addition to sym, dev, tr and $\operatorname{skw}\tau = \tau - \operatorname{sym}\tau$, we use the algebraic operation $S : \mathbb{M} \to \mathbb{M}$ defined by $S\tau = \tau^\top - \operatorname{tr}(\tau)\mathbb{I}$, whose inverse can be easily computed to be

$$S^{-1}\tau = \tau^\top - \frac{1}{2}\operatorname{tr}(\tau)\mathbb{I}.$$

We often use the summation convention and the alternating symbol $\varepsilon^{ijk} \equiv \varepsilon_{ijk}$ whose value equals $+1, -1$, or 0 according to whether ijk is a even, odd or not a permutation of 1, 2, 3. Using Cartesian unit vectors $e_i \equiv e^i$ and the summation convention, we write a vector v as $v = v_i e^i$. Using ε, one can express an isomorphism between skew-symmetric matrices in $\mathbb{K} = \operatorname{skw}\mathbb{M}$ and their axial vectors in $\mathbb{V}$, given by $\operatorname{mskw} : \mathbb{V} \to \mathbb{K}$, $\operatorname{mskw}(v^i e_i) = -\varepsilon^{ijk} v_k e_i \otimes e_j$. Let $\operatorname{vskw} : \mathbb{M} \to \mathbb{V}$ be defined by $\operatorname{vskw} = \operatorname{mskw}^{-1} \circ \operatorname{skw}$. For distributional fields w, vector fields v and matrix fields τ on three-dimensional domains, it is easy to see that the following identities hold:

$$\operatorname{div}\operatorname{mskw} v = -\operatorname{curl} v, \tag{26a}$$
$$\operatorname{mskw}\operatorname{grad} w = -\operatorname{curl}(w\mathbb{i}), \tag{26b}$$
$$\operatorname{mskw}\operatorname{curl} v = 2\operatorname{skw}\operatorname{grad} v, \tag{26c}$$
$$2\operatorname{skw}\operatorname{curl}\tau = \operatorname{mskw}\operatorname{div} S\tau, \tag{26d}$$
$$S\operatorname{grad} v = -\operatorname{curl}\operatorname{mskw} v, \tag{26e}$$
$$\operatorname{tr}\operatorname{curl}\tau = -2\operatorname{div}\operatorname{vskw}\tau, \tag{26f}$$

We start with two simple lemmas. Lemma 2.1 contains identities involving second-order partial differential operators and Lemma 2.2 gives density of the following smooth spaces with moment conditions:

$$\begin{aligned}
\mathcal{D}_{\mathbb{R}} &= \{\varphi \in \mathcal{D}(\Omega) : (\varphi, 1) = 0\},\\
\mathcal{D}_{\mathcal{RT}} &= \{\varphi \in \mathcal{D}(\Omega)\otimes\mathbb{V} : (\varphi, r) = 0 \text{ for all } r \in \mathcal{RT}\},\\
\mathcal{D}_{\mathcal{ND}} &= \{\varphi \in \mathcal{D}(\Omega)\otimes\mathbb{V} : (\varphi, r) = 0 \text{ for all } r \in \mathcal{ND}\},\\
\mathcal{D}_{\mathcal{P}_1} &= \{\varphi \in \mathcal{D}(\Omega) : (\varphi, p) = 0 \text{ for all } p \in \mathcal{P}_1\}.
\end{aligned} \tag{27}$$

For any two norms $\|\cdot\|_1$ and $\|\cdot\|_2$, we write

$$\|a\|_1 \lesssim \|b\|_2$$

to indicate that there is some constant $C > 0$ independent of a and b such that the inequality $\|a\|_1 \le C\|b\|_2$ holds.

Lemma 2.1. *The identities*

$$\operatorname{div}\top\operatorname{curl}\tau = \operatorname{curl}\operatorname{div}\top\tau, \tag{28}$$
$$\operatorname{curl}\top\operatorname{grad} u = \top\operatorname{grad}\operatorname{curl} u = \top\operatorname{dev}\operatorname{grad}\operatorname{curl} u, \tag{29}$$
$$\operatorname{div}\operatorname{sym}\operatorname{curl}\top\tau = \frac{1}{2}\operatorname{curl}\operatorname{div}\tau, \tag{30}$$
$$\operatorname{curl}\operatorname{def} u = \frac{1}{2}\top\operatorname{grad}\operatorname{curl} u = \frac{1}{2}\top\operatorname{dev}\operatorname{grad}\operatorname{curl} u, \tag{31}$$
$$\frac{1}{2}\operatorname{div}\top\operatorname{dev}\operatorname{grad} u = \frac{1}{3}\operatorname{grad}\operatorname{div} u \tag{32}$$

hold for any vector-valued distribution u and matrix-valued distribution τ.

Proof. To prove (28), we express row-wise curl using ε^{ijk}, the summation convention, and standard Cartesian unit vectors e_i,

$$\begin{aligned}
\operatorname{div}\top\operatorname{curl}\tau &= e_i\partial_j[\top\operatorname{curl}\tau]^{ij} = e_i\partial_j[\operatorname{curl}\tau]^{ji} = e_i\partial_j\varepsilon^{ikl}\partial_k\tau_{jl} = e_i\varepsilon^{ikl}\partial_k\partial_j[\top\tau]_{lj}\\
&= e_i\varepsilon^{ikl}\partial_k[\operatorname{div}\top\tau]_l = \operatorname{curl}\operatorname{div}\top\tau.
\end{aligned}$$

The first equality in (29) is proved similarly. For the second equality in (29), it suffices to note that the (i,j)th component of the matrix field $\operatorname{grad}\operatorname{curl} u$ equals $\partial_i\varepsilon^{jkl}\partial_k u_l$, so its trace, obtained with $i = j$ in this expression, vanishes.

Identity (30) follows using $\operatorname{div}\circ\operatorname{curl} = 0$ and (28):

$$\begin{aligned}\operatorname{div}\operatorname{sym}\operatorname{curl}\top\tau &= \operatorname{div}\frac{1}{2}\big(\operatorname{curl}\top\tau + \top\operatorname{curl}\top\tau\big)\\ &= \frac{1}{2}\operatorname{div}\top\operatorname{curl}\top\tau = \frac{1}{2}\operatorname{curl}\operatorname{div}\top\top\tau.\end{aligned}$$

Equation (31) follows from (29) and $\operatorname{curl}\circ\operatorname{grad} = 0$ in an analogous fashion. The proof of (32) using analogous techniques is also elementary. □

Lemma 2.2. *The spaces in* (27)*, namely* $\mathcal{D}_{\mathbb{R}}$*,* $\mathcal{D}_{\mathcal{RT}}$*,* $\mathcal{D}_{\mathcal{ND}}$*, and* $\mathcal{D}_{\mathcal{P}_1}$*, are dense in* $\mathring{\tilde{H}}^s_{\mathbb{R}}$ $\mathring{\tilde{H}}^s_{\mathcal{RT}}(\mathrm{curl})$*,* $\mathring{\tilde{H}}^s_{\mathcal{ND}}(\mathrm{div})$*, and* $\mathring{\tilde{H}}^s_{\mathcal{P}_1}$*, respectively, for any* $s \in \mathbb{R}$*.*

Proof. The proofs of all the four stated density results are similar. We only detail the second. Fix a nontrivial scalar function $b(x) \in \mathcal{D}(\Omega)$ satisfying $b(x) \ge 0$. Let ρ_i be a basis of the four-dimensional space $\mathcal{RT}$, normalized so that

$$(b\rho_i, \rho_j) = \delta_{ij} \tag{33}$$

where δ_{ij} denotes the Kronecker delta symbol. Let $v \in \mathring{\tilde{H}}^s_{\mathcal{RT}}(\mathrm{curl})$. In view of (21), we can find a sequence $\varphi_n \in \mathcal{D}(\Omega)\otimes\mathbb{V}$ such that

$$\lim_{n\to\infty} \|\varphi_n - v\|_{\mathring{\tilde{H}}^s(\mathrm{curl})} = 0. \tag{34}$$

Let

$$\psi_n(x) = \varphi_n(x) - \sum_{i=1}^{4} (\varphi_n, \rho_i)\, b(x)\, \rho_i(x). \tag{35}$$

Then ψ_n is in $\mathcal{D}(\Omega)\otimes\mathbb{V}$ and $(\psi_n, \rho_j) = 0$ due to (33), i.e., $\psi_n \in \mathcal{D}_{\mathcal{RT}}$. Moreover, ψ_n converges to v in $\mathring{\tilde{H}}^s(\mathrm{curl})$ as we now show: indeed, since $v(\rho_i) = 0$,

$$(\varphi_n, \rho_i) = (\varphi_n - v)(\rho_i) \le \|\varphi_n - v\|_{\mathring{\tilde{H}}^s}\|\rho_i\|_{H^{-s}}$$

by (19). Hence (34) implies that

$$\lim_{n\to\infty} (\varphi_n, \rho_i) = 0 \tag{36}$$

for each ρ_i. Now it is evident from (35) that ψ_n converges to v in $\mathring{\tilde{H}}^s$ norm since φ_n does. Moreover,

$$\operatorname{curl}(\psi_n - v) = \operatorname{curl}(\varphi_n - v) - \sum_{i=1}^{4} (\varphi_n, \rho_i)\ \operatorname{curl}(b\,\rho_i),$$

where, on the right hand side, the first term converges to zero in $\mathring{\tilde{H}}^s$ by (34), and the second term converges to zero by (36). Thus ψ_n and $\operatorname{curl}\psi_n$ converges in $\mathring{\tilde{H}}^s$ to v and $\operatorname{curl} v$, respectively. Hence $\mathcal{D}_{\mathcal{RT}}$ is dense in $\mathring{\tilde{H}}^s_{\mathcal{RT}}(\mathrm{curl})$. □

Theorem 2.3. *The diagram* (25) *commutes and every differential operator in it maps continuously (with respect to the norms of the indicated domains and codomains).*

Proof. Commutativity. The commutativity of the diagram cells in (row, column)-positions $(2, 3), (1, 2)$, and $(1, 3)$ follows respectively from identities (30), (31), and (32) of Lemma 2.1. The commutativity at positions across the diagonal also follow from these. At the remaining positions, it is obvious.

Next, let us prove the stated continuity properties. The continuity of the operators in the first row and column is standard. For the remaining operators, we use (18) and the following steps. We begin the maps in the second row of (25).

Continuity of $\operatorname{def} : \mathring{H}(\operatorname{curl}) \to \mathring{\widetilde{H}}_{\mathrm{cc}}$. For any $u \in \mathcal{D}(\Omega) \otimes \mathbb{V} \subset \mathring{H}(\operatorname{curl})$, note that $g = \operatorname{def} u \in \mathcal{D}(\Omega) \otimes \mathbb{S} \subset \mathring{\widetilde{H}}_{\mathrm{cc}}$ satisfies

$$\operatorname{curl} g = \frac{1}{2} \top \operatorname{dev} \operatorname{grad} \operatorname{curl} u = \frac{1}{2} \top \operatorname{grad} \operatorname{curl} u$$

due to (31) and (29). This implies that $\operatorname{inc} g := \operatorname{curl} \top \operatorname{curl} g = 0$. Hence, using (18),

$$\begin{aligned}\| \operatorname{def} u\|^2_{\mathring{\widetilde{H}}_{\mathrm{cc}}} &= \|g\|^2_{\mathring{\widetilde{H}}^{-1}} + \| \operatorname{curl} g\|^2_{\mathring{\widetilde{H}}^{-1}} + \| \operatorname{inc} g\|^2_{\mathring{\widetilde{H}}^{-1}} \\ &= \frac{1}{2}\| \operatorname{grad} u\|^2_{\mathring{\widetilde{H}}^{-1}} + \frac{1}{4}\| \operatorname{grad} \operatorname{curl} u\|^2_{\mathring{\widetilde{H}}^{-1}} \lesssim \|u\|^2_{H(\operatorname{curl})},\end{aligned}$$

which proves the continuity of the deformation operator by density.

Continuity of $\operatorname{curl} : \mathring{\widetilde{H}}_{\mathrm{cc}} \to \mathring{\widetilde{H}}_{\mathrm{cd}}$. It suffices to observe that $\tau = \operatorname{curl} g$, for any $g \in \mathcal{D}(\Omega) \otimes \mathbb{S} \subset \mathring{\widetilde{H}}_{\mathrm{cc}}$, satisfies $\tau = \operatorname{dev} \tau \in \mathcal{D}(\Omega) \otimes \mathbb{T} \subset \mathring{\widetilde{H}}_{\mathrm{cd}}$ and

$$\operatorname{div} \tau = 0, \qquad \operatorname{curl} \operatorname{div} \tau = 0, \qquad \operatorname{sym} \operatorname{curl} \top \tau = \operatorname{inc} g.$$

This shows that $\|\tau\|_{\mathring{\widetilde{H}}_{\mathrm{cd}}} \lesssim \|g\|_{\mathring{\widetilde{H}}_{\mathrm{cc}}}$ and the continuity follows by density.

Continuity of $\operatorname{div} : \mathring{\widetilde{H}}_{\mathrm{cd}} \to \mathring{\widetilde{H}}^{-1}_{\mathcal{RT}}(\operatorname{curl})$. Let $\tau \in \mathcal{D}(\Omega) \otimes \mathbb{T} \subset \mathring{\widetilde{H}}_{\mathrm{cd}}$. Then, for any $r = a + bx \in \mathcal{RT}$, $a \in \mathbb{V}, b \in \mathbb{R}$, we have

$$(\operatorname{div} \tau, r) = -(\tau, \operatorname{grad} r) = -(\tau, b\mathbb{i}) = 0$$

since $\tau : \mathbb{i}$ vanishes for $\tau(x) \in \mathbb{T}$. Next, by the definition of $\mathring{\widetilde{H}}_{\mathrm{cd}}$ norm,

$$\| \operatorname{div} \tau\|^2_{\mathring{\widetilde{H}}^{-1}} + \| \operatorname{curl} \operatorname{div} \tau\|^2_{\mathring{\widetilde{H}}^{-1}} \le \|\tau\|^2_{\mathring{\widetilde{H}}_{\mathrm{cd}}}.$$

Since the left hand side equals $\| \operatorname{div} \tau\|^2_{\mathring{\widetilde{H}}^{-1}(\operatorname{curl})}$, the continuity follows by density.

Continuity of $\operatorname{dev} \operatorname{grad} : \mathring{H}(\operatorname{div}) \to \mathring{\widetilde{H}}_{\mathrm{cd}\top}$. Let $\tau = \operatorname{dev} \operatorname{grad} q$ for some $q \in \mathcal{D}(\Omega) \otimes \mathbb{V} \subset \mathring{H}(\operatorname{div})$. Apply (32) to get

$$\operatorname{div} \top \tau = \frac{2}{3} \operatorname{grad} \operatorname{div} q,$$

which implies $\operatorname{curl} \operatorname{div} \top \tau = 0$. Also, since

$$\operatorname{sym} \operatorname{curl} \tau = \operatorname{sym} \operatorname{curl} \frac{1}{3}(\operatorname{div} q)\mathbb{i},$$

all terms in the norm $\|\tau\|_{\mathring{\widetilde{H}}_{\mathrm{cd}\top}}$ can be bounded by the $\mathring{\widetilde{H}}^{-1}$-norms of the first order of derivatives of $\operatorname{div} q$ and q, so using (18), $\|\tau\|_{\mathring{\widetilde{H}}_{\mathrm{cd}\top}} \lesssim \|q\|_{H(\operatorname{div})}$ and the continuity follows by density.

Continuity of $\operatorname{sym} \operatorname{curl} \top : \mathring{\widetilde{H}}_{\mathrm{cd}} \to \mathring{\widetilde{H}}_{\mathrm{dd}}$. This is a bounded operator since $\sigma = \operatorname{sym} \operatorname{curl} \top \tau$ for any $\tau \in \mathcal{D}(\Omega) \otimes \mathbb{T} \subset \mathring{\widetilde{H}}_{\mathrm{cd}}$ satisfies, due to (30),

$$\operatorname{div} \sigma = \frac{1}{2} \operatorname{curl} \operatorname{div} \tau,$$

which in turn implies $\operatorname{div} \operatorname{div} \sigma = 0$. Thus $\|\sigma\|_{\mathring{\widetilde{H}}_{\mathrm{dd}}} \lesssim \|\tau\|_{\mathring{\widetilde{H}}_{\mathrm{cd}}}$.

Continuity of $\operatorname{div} : \mathring{\widetilde{H}}_{\mathrm{dd}} \to \mathring{\widetilde{H}}^{-1}_{\mathcal{ND}}(\operatorname{div})$. First note that for any $\sigma \in \mathcal{D}(\Omega) \otimes \mathbb{S} \subset \mathring{\widetilde{H}}_{\mathrm{dd}}$ and $r = a + b \times x \in \mathcal{ND}$, $a, b \in \mathbb{V}$,

$$(\operatorname{div} \sigma, r) = -(\sigma, \operatorname{grad} r) = 0$$

because $\sigma : \operatorname{grad} r = \sigma_{ij}\partial_j[b \times x]_i = \varepsilon^{ijp}\sigma_{ij}b_p$ vanishes due to the symmetry $\sigma_{ij} = \sigma_{ji}$. Thus $\operatorname{div}\sigma$ is in $\mathring{\tilde{H}}^{-1}_{\mathcal{ND}}(\operatorname{div})$. Moreover, by the definition of the $\mathring{\tilde{H}}_{\mathrm{dd}}$-norm

$$\|\operatorname{div}\sigma\|^2_{\mathring{\tilde{H}}^{-1}} + \|\operatorname{div}\operatorname{div}\sigma\|^2_{\mathring{\tilde{H}}^{-1}} \le \|\sigma\|^2_{\mathring{\tilde{H}}_{\mathrm{dd}}}.$$

The left hand side exactly equals $\|\operatorname{div}\sigma\|^2_{\mathring{\tilde{H}}^{-1}(\mathrm{div})}$ so the continuity follows by density.

Continuity of $\operatorname{grad} : L_{2,\mathbb{R}} \to \mathring{\tilde{H}}^{-1}_{\mathcal{RT}}(\operatorname{curl})$. Let $u \in \mathcal{D}(\Omega) \cap L_{2,\mathbb{R}} = \mathcal{D}_{\mathbb{R}}$. By Lemma 2.2 $\mathcal{D}_{\mathbb{R}}$ is dense in $L_{2,\mathbb{R}}$. For any $r = a + bx \in \mathcal{RT}$, $a \in \mathbb{V}, b \in \mathbb{R}$, integrating by parts using the compact support of u,

$$(\operatorname{grad} u, r) = -(u, \operatorname{div} r) = -3(u, b) = 0$$

since u has zero mean value on Ω. Hence $\operatorname{grad} u$ is in $\mathring{\tilde{H}}^{-1}_{\mathcal{RT}}(\operatorname{curl})$, so by Lemma 2.2, $\operatorname{grad} L_{2,\mathbb{R}} \subseteq \mathring{\tilde{H}}^{-1}_{\mathcal{RT}}(\operatorname{curl})$. The needed boundedness estimate is immediate from (18).

Continuity of $\operatorname{curl} : \mathring{\tilde{H}}^{-1}_{\mathcal{RT}}(\operatorname{curl}) \to \mathring{\tilde{H}}^{-1}_{\mathcal{ND}}(\operatorname{div})$. By Lemma 2.2, $\mathcal{D}_{\mathcal{RT}}$ is dense in $\mathring{\tilde{H}}^{-1}_{\mathcal{RT}}(\operatorname{curl})$. Let $v \in \mathcal{D}_{\mathcal{RT}}$. Then, given any $r = a + b \times x \in \mathcal{ND}$ with some $a, b \in \mathbb{V}$,

$$(\operatorname{curl} v, r) = (v, \operatorname{curl} r) = 2(v, b) = 0$$

since v is orthogonal to $\mathcal{RT}$. Hence $\operatorname{curl} v$ is in $\mathring{\tilde{H}}^{-1}_{\mathcal{ND}}(\operatorname{div})$. Combined with the obvious norm bound, the continuity follows from density.

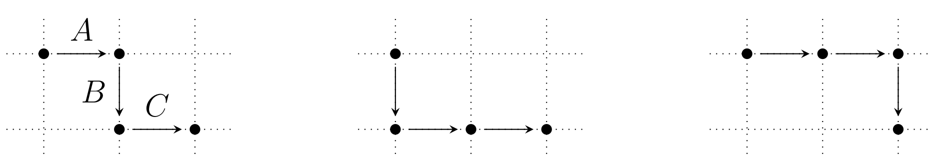
Continuity of $\operatorname{div} : \mathring{\tilde{H}}^{-1}_{\mathcal{ND}}(\operatorname{div}) \to \mathring{\tilde{H}}^{-1}_{\mathcal{P}_1}$. Let $q \in \mathcal{D}_{\mathcal{ND}}$. Then, for any $p \in \mathcal{P}_1$,

$$(\operatorname{div} q, p) = -(q, \operatorname{grad} p) = 0$$

since $\operatorname{grad} p$ is in $\mathcal{ND}$ and q is orthogonal to $\mathcal{ND}$. Therefore $\operatorname{div} q$ is in $\mathring{\tilde{H}}^{-1}_{\mathcal{P}_1}$. In view of the obvious norm bound, the density of $\mathcal{D}_{\mathcal{ND}}$ in $\mathring{\tilde{H}}^{-1}_{\mathcal{ND}}(\operatorname{div})$ (given by Lemma 2.2) finishes the proof. ☐

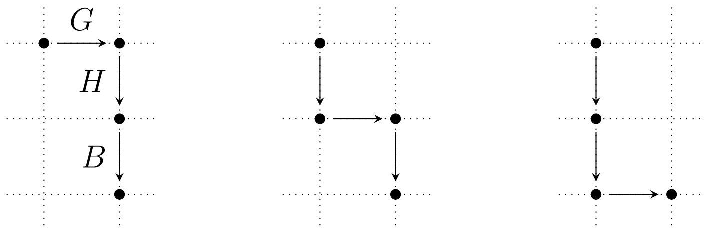
Theorem 2.4. *All paths in the diagram* (25) *are 2-complexes.*

Proof. Consider paths of the following form:



If the first path above is a 2-complex, then by the commutativity properties of Theorem 2.3, the second and third are also 2-complexes. Proving that the first path is a 2-complex, i.e., showing that $C \circ B \circ A = 0$ for all such A, B, C in (25), is (tedious but) elementary using the identities of (26) and Lemma 2.1. For example, with $A = \operatorname{grad} : \mathring{H}(\operatorname{grad}) \to \mathring{H}(\operatorname{curl}), B = \operatorname{def} : \mathring{H}(\operatorname{curl}) \to \mathring{\tilde{H}}_{\mathrm{cc}}$, and $C = \operatorname{curl} : \mathring{\tilde{H}}_{\mathrm{cc}} \to \mathring{\tilde{H}}_{\mathrm{cd}}$, we have $C \circ B \circ A = \operatorname{curl} \circ \frac{1}{2}(\operatorname{grad} + \top \operatorname{grad}) \circ \operatorname{grad} = \frac{1}{2} \operatorname{curl} \circ \top \operatorname{grad} \circ \operatorname{grad} = \frac{1}{2} \operatorname{curl} \circ \top \operatorname{grad} \circ \operatorname{grad} = \frac{1}{2} \top \operatorname{grad} \circ \operatorname{curl} \circ \operatorname{grad} = 0$, where we have used the identity (29) of Lemma 2.1. Similarly, it is elementary to see that $B \circ H \circ G = 0$

in paths taken from (25) of the form in the first diagram below,



so by commutativity, all paths of the three types shown above are also 2-complexes. These types of paths exhaust all possibilities. □

Next, we consider the fundamental *second-order differential operators* inherent in (25): the incompatibility operator defined in (22), the hessian operator $\mathrm{hess} := \mathrm{def} \circ \mathrm{grad}$, $\mathrm{grad} \circ \mathrm{div}$, $\mathrm{curl} \circ \mathrm{div}$ and $\mathrm{div} \circ \mathrm{div}$. These appear along the diagonals of the following diagram:

$$
\begin{array}{ccccccc}
\mathring{H}(\mathrm{grad}) & \xrightarrow{\mathrm{grad}} & \mathring{H}(\mathrm{curl}) & \xrightarrow{\mathrm{curl}} & \mathring{H}(\mathrm{div}) & \xrightarrow{\mathrm{div}} & L_{2,\mathbb{R}} \\
\downarrow{\scriptstyle \mathrm{grad}} & \searrow^{\mathrm{hess}} & \downarrow{\scriptstyle \mathrm{def}} & \searrow^{\mathrm{curl\,def}} & \downarrow{\scriptstyle \frac{1}{2}\top\,\mathrm{dev\,grad}} & \searrow^{\frac{1}{3}\mathrm{grad\,div}} & \downarrow{\scriptstyle \frac{1}{3}\mathrm{grad}} \\
\mathring{H}(\mathrm{curl}) & \xrightarrow{\mathrm{def}} & \mathring{\widetilde{H}}_{\mathrm{cc}} & \xrightarrow{\mathrm{curl}} & \mathring{\widetilde{H}}_{\mathrm{cd}} & \xrightarrow{\mathrm{div}} & \mathring{\widetilde{H}}^{-1}_{\mathcal{RT}}(\mathrm{curl}) \\
\downarrow{\scriptstyle \mathrm{curl}} & & \downarrow{\scriptstyle \top\,\mathrm{curl}} & \searrow^{\mathrm{inc}} & \downarrow{\scriptstyle \mathrm{sym\,curl}\,\top} & \searrow^{\frac{1}{2}\mathrm{curl\,div}} & \downarrow{\scriptstyle \frac{1}{2}\mathrm{curl}} \\
\mathring{H}(\mathrm{div}) & \xrightarrow{\frac{1}{2}\mathrm{dev\,grad}} & \mathring{\widetilde{H}}_{\mathrm{cd}\top} & \xrightarrow{\mathrm{sym\,curl}} & \mathring{\widetilde{H}}_{\mathrm{dd}} & \xrightarrow{\mathrm{div}} & \mathring{\widetilde{H}}^{-1}_{\mathcal{ND}}(\mathrm{div}) \\
\downarrow{\scriptstyle \mathrm{div}} & & \downarrow{\scriptstyle \mathrm{div}\,\top} & & \downarrow{\scriptstyle \mathrm{div}} & \searrow^{\mathrm{div\,div}} & \downarrow{\scriptstyle \mathrm{div}} \\
L_{2,\mathbb{R}} & \xrightarrow{\frac{1}{3}\mathrm{grad}} & \mathring{\widetilde{H}}^{-1}_{\mathcal{RT}}(\mathrm{curl}) & \xrightarrow{\frac{1}{2}\mathrm{curl}} & \mathring{\widetilde{H}}^{-1}_{\mathcal{ND}}(\mathrm{div}) & \xrightarrow{\mathrm{div}} & \mathring{\widetilde{H}}^{-1}_{\mathcal{P}_1}
\end{array}
\tag{37}
$$

The second-order operators below the diagonal are not shown as they are mirrored by those shown above the diagonal. Note that each indicated second-order operator is a composition of adjacent first-order operators at the top and right, or by commutativity, the adjacent left and bottom operators. We can now read off less regular versions of (5), (6), and (7) from the diagram (37), as stated next.

Corollary 2.5. *The following paths in* (37) *are complexes:*

(1) *The hessian complex:*

$$
\begin{array}{ccccccc}
\mathring{H}(\mathrm{grad}) & & & & & & \\
 & \searrow^{\mathrm{hess}} & & & & & \\
 & & \mathring{\widetilde{H}}_{\mathrm{cc}} & \xrightarrow{\mathrm{curl}} & \mathring{\widetilde{H}}_{\mathrm{cd}} & \xrightarrow{\mathrm{div}} & \mathring{\widetilde{H}}^{-1}_{\mathcal{RT}}(\mathrm{curl})
\end{array}
\tag{38}
$$

(2) *The elasticity complex:*

$$
\begin{array}{ccccccc}
\mathring{H}(\mathrm{curl}) & \xrightarrow{\mathrm{def}} & \mathring{\widetilde{H}}_{\mathrm{cc}} & & & & \\
 & & & \searrow^{\mathrm{inc}} & & & \\
 & & & & \mathring{\widetilde{H}}_{\mathrm{dd}} & \xrightarrow{\mathrm{div}} & \mathring{\widetilde{H}}^{-1}_{\mathcal{ND}}(\mathrm{div})
\end{array}
\tag{39}
$$

(3) *The* div div *complex:*

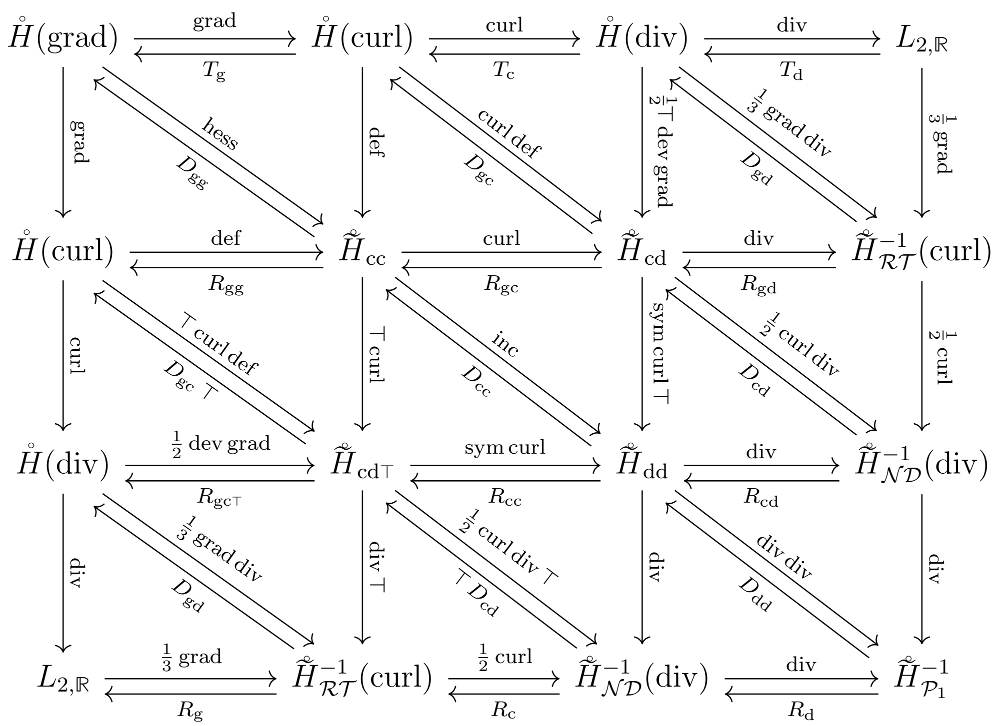
$$
\begin{array}{cccccccc}
\mathring{H}(\mathrm{div}) & \xrightarrow{\frac{1}{2}\,\mathrm{dev}\,\mathrm{grad}} & \mathring{\widetilde{H}}_{\mathrm{cd}^\top} & \xrightarrow{\mathrm{sym}\,\mathrm{curl}} & \mathring{\widetilde{H}}_{\mathrm{dd}} & & \\
 & & & & & \searrow^{\mathrm{div}\,\mathrm{div}} & \\
 & & & & & & \mathring{\widetilde{H}}^{-1}_{\mathcal{P}_1}
\end{array}
\tag{40}
$$

Proof. These statements follow from the 2-complex properties of Theorem 2.4 and elementary manipulations with first-order differential operators. For instance, to prove the last, $\mathrm{sym}\,\mathrm{curl} \circ \mathrm{dev}\,\mathrm{grad}\, w = -\frac{1}{3}\,\mathrm{sym}\,\mathrm{curl}(\mathrm{tr}(\mathrm{grad}\, w)\mathring{\mathbb{I}}) = -\frac{1}{3}\,\mathrm{sym}\,\mathrm{mskw}\,\mathrm{grad}(\mathrm{tr}(\mathrm{grad}\, w)) = 0$, where we have used (26b); and of course, $\mathrm{div}\,\mathrm{div} \circ \mathrm{sym}\,\mathrm{curl}$ must vanish due to the 2-complex property of Theorem 2.4. □

3. Regular decompositions, density, and continuous right inverses

Right inverses of exterior derivatives that are continuous in appropriate Sobolev norms were given in [16], inspired by the classical work of [10]. In this section, we leverage their results to show regular decompositions of some Sobolev spaces of matrix fields, prove density of smooth functions in them, and construct right inverses of the differential operators in (37).

The right inverses of the derivative operators in (37) that act on or produce matrix fields are the subscripted D and R operators labeling the diagonally upward arrows and rightward arrows in the following diagram:



(41)

The downward arrows in (41) can also be provided with corresponding upward right inverses. They are not marked to reduce clutter and because the same information is contained in the horizontal arrows back and forth. Below, we detail the construction of each new right inverse operator. Note that some of the D and R operators map from and into *subspaces* of the respective domains and codomains indicated in (41). The codomain subspaces consist of (more) *regular* functions. The domain subspaces are *kernels* of one of the (multiple) operators acting on the space. These subspaces are given precisely in the case by case results below (and summarized in Theorem 3.21). Throughout, we denote the null space of a linear operator $A : X \to Y$ by

$$\ker(A : X).$$

Note, e.g., $\ker(\operatorname{curl} : \mathring{H}(\operatorname{curl}))$ is different from $\ker(\operatorname{curl} : \mathring{\widetilde{H}}_{\mathrm{cc}}) = \{g \in \mathring{\widetilde{H}}_{\mathrm{cc}} : \operatorname{curl} g = 0\}$.

Regular decompositions for standard Sobolev spaces based on the exterior derivative can be inferred from the results of [16]. In this section, we also show how to combine their results with our previous results to produce regular decompositions for the following new spaces of matrix fields:

$$\begin{aligned}
\hat{H}_{\mathrm{cc}} &= \{g \in \mathring{\widetilde{H}}^{-1} \otimes \mathbb{S} : \operatorname{curl} g \in \mathring{\widetilde{H}}^{-1} \otimes \mathbb{V}, \operatorname{inc} g \in \mathring{\widetilde{H}}^{-1} \otimes \mathbb{S}\},\\
\hat{H}_{\mathrm{cd}} &= \{\tau \in \mathring{\widetilde{H}}^{-1} \otimes \mathbb{T} : \operatorname{div} \tau \in \mathring{\widetilde{H}}^{-1} \otimes \mathbb{V}, \operatorname{sym}\operatorname{curl} \tau \in \mathring{\widetilde{H}}^{-1} \otimes \mathbb{S}, \operatorname{curl}\operatorname{div} \tau \in \mathring{\widetilde{H}}^{-1} \otimes \mathbb{V}\},\\
\hat{H}_{\mathrm{dd}} &= \{\sigma \in \mathring{\widetilde{H}}^{-1} \otimes \mathbb{S} : \operatorname{div} \sigma \in \mathring{\widetilde{H}}^{-1} \otimes \mathbb{V}, \operatorname{div}\operatorname{div} \sigma \in \mathring{\widetilde{H}}^{-1}\}.
\end{aligned}$$

They are normed, respectively, by $\|\cdot\|_{\hat{H}_{\mathrm{cc}}}, \|\cdot\|_{\hat{H}_{\mathrm{cd}}}, \|\cdot\|_{\hat{H}_{\mathrm{dd}}}$, the norms defined in (23). Obviously, the spaces defined in (24) are subspaces of these spaces, i.e.,

$$\mathring{\widetilde{H}}_{\mathrm{cc}} \subseteq \hat{H}_{\mathrm{cc}}, \quad \mathring{\widetilde{H}}_{\mathrm{cd}} \subseteq \hat{H}_{\mathrm{cd}}, \quad \mathring{\widetilde{H}}_{\mathrm{dd}} \subseteq \hat{H}_{\mathrm{dd}}. \tag{42}$$

The theorems in this section (Theorems 3.4, 3.7, and 3.10) improve these inclusions to equalities, thus also proving the density of their respective subspaces of compactly supported smooth functions.

Our results are under the additional assumptions on Ω that it is simply connected and that its boundary is connected. Then the topology of Ω is trivial. Covering Ω by subdomains starlike with respect to a ball and using regularized Bogovskiĭ operators in each subdomain, [16, Theorem 4.9] proves that there exist continuous linear operators

$$T_{\rm g} : \mathring{\tilde{H}}^s \otimes \mathbb{V} \to \mathring{\tilde{H}}^{s+1}, \qquad T_{\rm c} : \mathring{\tilde{H}}^s \otimes \mathbb{V} \to \mathring{\tilde{H}}^{s+1} \otimes \mathbb{V}, \qquad T_{\rm d} : \mathring{\tilde{H}}^s \to \mathring{\tilde{H}}^{s+1}, \tag{43a}$$

satisfying

$$\operatorname{grad}(T_{\rm g} v) = v \qquad \text{for all } v \in \mathring{\tilde{H}}^s \otimes \mathbb{V} \text{ with } \operatorname{curl} v = 0, \tag{43b}$$

$$\operatorname{curl}(T_{\rm c} q) = q \qquad \text{for all } q \in \mathring{\tilde{H}}^s \otimes \mathbb{V} \text{ with } \operatorname{div} q = 0, \tag{43c}$$

$$\operatorname{div}(T_{\rm d} u) = u \qquad \text{for all } u \in \mathring{\tilde{H}}^s \text{ with } u(1) = 0, \tag{43d}$$

for any real number s, where $\mathring{\tilde{H}}^s$ is the subspace of distributions on $\mathbb{R}^3$ defined in (17). The last condition in (43d) is a zero mean condition on u given through a functional action that makes sense even for negative s. For $s \geq 0$, it can be expressed using the L_2 inner product as $(u, 1) = 0$.

It is standard to use (43) to produce regular decompositions of $\mathring{H}(\text{curl})$ and $\mathring{H}(\text{div})$. Indeed, any $u \in \mathring{H}(\text{curl})$ can be decomposed into

$$u = \mathring{S}_{\rm c}^{(0)} u + \operatorname{grad} \mathring{S}_{\rm c}^{(1)} u, \quad \text{ with } \mathring{S}_{\rm c}^{(0)} u = T_{\rm c} \operatorname{curl} u,\ \mathring{S}_{\rm c}^{(1)} u = T_{\rm g}(u - T_{\rm c} \operatorname{curl} u), \tag{44}$$

as can be immediately verified using (43c) and (43b). By the continuity properties of $T_{\rm g}$ and $T_{\rm c}$, the operators $\mathring{S}_{\rm c}^{(0)} : \mathring{H}(\text{curl}) \to \mathring{H}^1 \otimes \mathbb{V}$ and $\mathring{S}_{\rm c}^{(1)} : \mathring{H}(\text{curl}) \to \mathring{H}^1$ are continuous. Since $\mathring{S}_{\rm c}^{(0)} u$ and $\mathring{S}_{\rm c}^{(1)} u$ have $\mathring{H}^1$-regularity (higher than what may be expected of u), this is referred to as a "regular decomposition" of $\mathring{H}(\text{curl})$. The process of arriving at this decomposition can be viewed as first generating a zero curl function $u - T_{\rm c} \operatorname{curl} u$ and then moving *left* of $\mathring{H}(\text{curl})$ in the diagram (41) to create a potential in $\mathring{H}^1$ using the operator $T_{\rm g}$. For the matrix-valued function spaces in the middle of (41), the process is similar, but we have more options to move, such as *up, left, or diagonally,* and our regular decompositions that follow have multiple potentials.

3.1. Regular decomposition of $\mathring{\tilde{H}}_{\rm cc}$.

We start with a result that can also be found in [2, Theorem 2] as a special case of existence of regular potential. Here we provide an explicit construction (see also [11]).

Lemma 3.1. *There is a linear map* $D_{\rm cc} : \ker(\operatorname{div} : \mathring{\tilde{H}}^s \otimes \mathbb{S}) \to \mathring{\tilde{H}}^{s+2} \otimes \mathbb{S}$ *such that*

$$\sigma = \operatorname{inc} D_{\rm cc}\sigma, \qquad \|D_{\rm cc}\sigma\|_{\mathring{\tilde{H}}^{s+2}} \lesssim \|\sigma\|_{\mathring{\tilde{H}}^s}.$$

for any $s \in \mathbb{R}$.

Proof. Let $\sigma \in \mathring{\tilde{H}}^s \otimes \mathbb{S}$ and $\operatorname{div} \sigma = 0$. Applying $T_{\rm c}$ to row vectors of σ, whose components are all distributions in $\mathbb{R}^3$ supported on $\bar{\Omega}$ per (17), we find from (43c) that there is a $\eta \in \mathring{\tilde{H}}^{s+1} \otimes \mathbb{M}$ such that the identity

$$\sigma = \operatorname{curl} \eta$$

holds in $\mathbb{R}^3$. Also, since σ is symmetric,

$$\operatorname{skw}\sigma = 0 = \operatorname{skw}\operatorname{curl}\eta = \frac{1}{2}\operatorname{mskw}\operatorname{div} S\eta$$

by (26d). Hence $S\eta \in \mathring{\widetilde{H}}^{s+1} \otimes \mathbb{M}$ has vanishing divergence in all $\mathbb{R}^3$ (and obviously $S\eta$ is supported on $\bar{\Omega}$). Applying T_{c} row-wise to $S\eta$, we conclude from (43c) that there is a $\gamma \in \mathring{\widetilde{H}}^{s+2} \otimes \mathbb{M}$ such that

$$S\eta = \operatorname{curl}\gamma.$$

Set $g = \operatorname{sym}\gamma$ in $\mathring{\widetilde{H}}^{s+2} \otimes \mathbb{S}$. Then

$$\begin{aligned}\sigma &= \operatorname{curl}\eta = \operatorname{curl} S^{-1}\operatorname{curl}\gamma \\ &= \operatorname{curl} S^{-1}\operatorname{curl}(\operatorname{skw}\gamma + g) = \operatorname{inc} g\end{aligned}$$

where the last equality is due to (26e) and (26f). By the continuity of T_{c}, the linear map $\sigma \mapsto g$ we just constructed is continuous and is the needed map D_{cc}. □

Lemma 3.2. *There is a linear map* $\tilde{R}_{\mathrm{gg}} : \ker(\operatorname{inc} : \hat{H}_{\mathrm{cc}}) \to \mathring{H}^1 \otimes \mathbb{V}$ *such that for any* $g \in \ker(\operatorname{inc} : \hat{H}_{\mathrm{cc}})$,

$$\operatorname{curl}\operatorname{def}\tilde{R}_{\mathrm{gg}} g = \operatorname{curl} g, \qquad \|\tilde{R}_{\mathrm{gg}} g\|_{H^1} \lesssim \|\operatorname{curl} g\|_{\mathring{\widetilde{H}}^{-1}}.$$

Proof. Given any $g \in \ker(\operatorname{inc} : \hat{H}_{\mathrm{cc}})$, since $\operatorname{inc} g = \operatorname{curl}(\top\operatorname{curl} g) = 0$, applying T_{g} to each row vector of $\top\operatorname{curl} g$ in $\mathring{\widetilde{H}}^{-1} \otimes \mathbb{V}$ and using (43b), we obtain a $q \in L_2(\mathbb{R}^3) \otimes \mathbb{V}$ satisfying

$$\top\operatorname{curl} g = \operatorname{grad} q$$

on all $\mathbb{R}^3$. Moreover, by (26f), $\operatorname{curl} g$ has zero trace, so

$$0 = \operatorname{tr}(\top\operatorname{curl} g) = \operatorname{tr}(\operatorname{grad} q) = \operatorname{div} q.$$

Hence $u = \frac{1}{2}T_{\mathrm{c}} q$ is in $\mathring{H}^1 \otimes \mathbb{V}$ and satisfies $q = \frac{1}{2}\operatorname{curl} u$. Therefore, using (31),

$$\operatorname{curl} g = \top\operatorname{grad} q = \frac{1}{2}\top\operatorname{grad}\operatorname{curl} u = \operatorname{curl}\operatorname{def} u.$$

Denoting the linear map $g \mapsto u$ by $\tilde{R}_{\mathrm{gg}}$, the proof is now completed using the continuity of T_{g} and T_{c}. □

Lemma 3.3. *There is a linear map* $D_{\mathrm{gg}} : \ker(\operatorname{curl} : \mathring{\widetilde{H}}^s \otimes \mathbb{S}) \to \mathring{\widetilde{H}}^{s+2}$ *such that*

$$g = \operatorname{hess} D_{\mathrm{gg}}\, g, \qquad \|\, D_{\mathrm{gg}}\, g\|_{\mathring{\widetilde{H}}^{s+2}} \lesssim \|g\|_{\mathring{\widetilde{H}}^{s}}.$$

for any $s \in \mathbb{R}$ *and* $g \in \mathring{\widetilde{H}}^s$ *with vanishing curl.*

Proof. Let $g \in \mathring{\widetilde{H}}^s$ have zero curl. Then applying T_{g} to the row vectors of g and using (43b), we obtain a $u \in \mathring{\widetilde{H}}^{s+1} \otimes \mathbb{V}$, supported on $\bar{\Omega}$, such that the identity

$$g = \operatorname{grad} u$$

holds on all $\mathbb{R}^3$. Applying sym to both sides, $g = \operatorname{def} u$. A further application of curl on both sides yields

$$0 = \operatorname{curl} g = \operatorname{curl}\operatorname{def} u = \frac{1}{2}\top\operatorname{grad}\operatorname{curl} u,$$

by (31), i.e., all first order derivatives of $\operatorname{curl} u$ vanish. Hence there must exist a constant vector $b \in \mathbb{V}$ such that $\operatorname{curl} u = b$ holds on all $\mathbb{R}^3$. But u is supported on $\bar{\Omega}$, so b must be the zero vector. Now that we have shown $\operatorname{curl} u = 0$, putting $w = T_{\mathrm{g}} u$ and using (43b), we find that $w \in \mathring{\widetilde{H}}^{s+2}$ satisfies $\operatorname{grad} w = u$ and

$$\operatorname{hess}(w) = \operatorname{def}(\operatorname{grad} w) = \operatorname{def}(u) = g.$$

The linear map $g \mapsto w$ we just constructed is the needed operator D_{gg}. □

Theorem 3.4. *There exist three continuous linear operators*

$$\mathring{S}^{(0)}_{\mathrm{cc}} : \hat{H}_{\mathrm{cc}} \to \mathring{H}^1 \otimes \mathbb{S}, \qquad \mathring{S}^{(1)}_{\mathrm{cc}} : \hat{H}_{\mathrm{cc}} \to \mathring{H}^1 \otimes \mathbb{V}, \qquad \mathring{S}^{(2)}_{\mathrm{cc}} : \hat{H}_{\mathrm{cc}} \to \mathring{H}^1,$$

such that any $g \in \hat{H}_{\mathrm{cc}}$ can be decomposed into

$$g = \mathring{S}^{(0)}_{\mathrm{cc}}\, g + \operatorname{def} \mathring{S}^{(1)}_{\mathrm{cc}}\, g + \operatorname{hess} \mathring{S}^{(2)}_{\mathrm{cc}}\, g. \tag{45}$$

Consequently, $\hat{H}_{\mathrm{cc}} = \mathring{\widetilde{H}}_{\mathrm{cc}}$.

Proof. Put $\mathring{S}^{(0)}_{\mathrm{cc}}\, g := D_{\mathrm{cc}} \operatorname{inc} g$. By Lemma 3.1,

$$\operatorname{inc}(g - \mathring{S}^{(0)}_{\mathrm{cc}}\, g) = 0.$$

Consequently, by Lemma 3.2, $\mathring{S}^{(1)}_{\mathrm{cc}}\, g := \tilde{R}_{\mathrm{gg}}(g - \mathring{S}^{(0)}_{\mathrm{cc}}\, g)$ is in $\mathring{H}^1 \otimes \mathbb{V}$ satisfies

$$\operatorname{curl}\big(g - \mathring{S}^{(0)}_{\mathrm{cc}}\, g - \operatorname{def} \mathring{S}^{(1)}_{\mathrm{cc}}\, g\big) = 0. \tag{46}$$

Applying Lemma 3.3 with $s = -1$, we find that $\mathring{S}^{(2)}_{\mathrm{cc}}\, g := D_{\mathrm{gg}}(g - \mathring{S}^{(0)}_{\mathrm{cc}}\, g - \operatorname{def} \mathring{S}^{(1)}_{\mathrm{cc}}\, g)$ satisfies

$$g - \mathring{S}^{(0)}_{\mathrm{cc}}\, g - \operatorname{def} \mathring{S}^{(1)}_{\mathrm{cc}}\, g = \operatorname{hess} \mathring{S}^{(2)}_{\mathrm{cc}}\, g,$$

and has the required continuity property, thus completing the proof of 45.

To prove that $\hat{H}_{\mathrm{cc}} = \mathring{\widetilde{H}}_{\mathrm{cc}}$, in view of (42), it suffices to prove that any $g \in \hat{H}_{\mathrm{cc}}$, decomposed as above into $g = \mathring{S}^{(0)}_{\mathrm{cc}}\, g + \operatorname{def} \mathring{S}^{(1)}_{\mathrm{cc}}\, g + \operatorname{hess} \mathring{S}^{(2)}_{\mathrm{cc}}\, g$, is in $\mathring{\widetilde{H}}_{\mathrm{cc}}$. By the density of $\mathcal{D}(\Omega)$ in $\mathring{H}^1(\Omega)$, there are $g_m \in \mathcal{D}(\Omega) \otimes \mathbb{S}$, $u_m \in \mathcal{D}(\Omega) \otimes \mathbb{V}$, and $w_m \in \mathcal{D}(\Omega)$ such that

$$\|g_m - \mathring{S}^{(0)}_{\mathrm{cc}}\, g\|_{H^1} \to 0, \qquad \|u_m - \mathring{S}^{(1)}_{\mathrm{cc}}\, g\|_{H^1} \to 0, \qquad \|w_m - \mathring{S}^{(2)}_{\mathrm{cc}}\, g\|_{H^1} \to 0,$$

as $m \to \infty$. Hence, by (18), $g_m + \operatorname{def} u_m + \operatorname{hess} w_m \in \mathcal{D}(\Omega) \otimes \mathbb{S}$ converges to g in $\|\cdot\|_{\mathring{\widetilde{H}}_{\mathrm{cc}}}$-norm, proving that $g \in \mathring{\widetilde{H}}_{\mathrm{cc}}$. □

3.2. **Regular decomposition of $\mathring{\widetilde{H}}_{\mathrm{dd}}$.**

Lemma 3.5. *There is a linear map $D_{\mathrm{dd}} : \mathring{\widetilde{H}}^s_{\mathcal{P}_1} \to \mathring{\widetilde{H}}^{s+2} \otimes \mathbb{S}$ such that for any $s \in \mathbb{R}$ and $w \in \mathring{\widetilde{H}}^s_{\mathcal{P}_1}$,*

$$\operatorname{div} \operatorname{div} D_{\mathrm{dd}} w = w, \qquad \|D_{\mathrm{dd}} w\|_{\mathring{\widetilde{H}}^{s+2}} \lesssim \|w\|_{\mathring{\widetilde{H}}^s}. \tag{47}$$

Proof. Let $w \in \mathcal{D}_{\mathcal{P}_1}$. Since $(w, 1) = 0$, by (43d), $q = T_{\mathrm{d}} w$ satisfies

$$\operatorname{div} q = w, \qquad \|q\|_{\mathring{\widetilde{H}}^{s+1}} \lesssim \|w\|_{\mathring{\widetilde{H}}^s}.$$

Since q is supported on $\bar{\Omega}$, we may integrate by parts to see that $0 = (w, p_1) = (\operatorname{div} q, p_1) = -(q, \operatorname{grad} p_1)$ for any $p_1 \in \mathcal{P}_1$. Thus all components of q have zero mean on Ω. Applying (43d) again, we then obtain a $\tau \in \mathring{\widetilde{H}}^{s+2} \otimes \mathbb{M}$ such that $\operatorname{div} \tau = q$. Let $u = \operatorname{vskw} \tau$. Then $\operatorname{div} \operatorname{skw} \tau = \operatorname{div} \operatorname{mskw} u = -\operatorname{curl} u$ by (26a). Collecting these observations, and putting $\sigma = \operatorname{sym} \tau$,

$$\begin{aligned} w = \operatorname{div} q = \operatorname{div} \operatorname{div} \tau &= \operatorname{div} \operatorname{div}(\operatorname{sym} \tau + \operatorname{skw} \tau) \\ &= \operatorname{div} \operatorname{div} \sigma - \operatorname{div} \operatorname{curl} u = \operatorname{div} \operatorname{div} \sigma. \end{aligned}$$

Denote the linear map $w \mapsto \sigma$ we just constructed by $D_{\mathrm{dd}}w$. By the continuity of T_{d}, we see that D_{dd} satisfies the norm estimate in (47) for all $w \in \mathcal{D}_{\mathcal{P}_1}$. Hence by the density result of Lemma 2.2, D_{dd} has a unique continuous extension, which is the required map. $\square$

Lemma 3.6. *There is a linear map* $\tilde{R}_{\mathrm{cc}} : \ker(\operatorname{div}\operatorname{div} : \hat{H}_{\mathrm{dd}}) \to \mathring{H}^1 \otimes \mathbb{T}$ *such that for all* $\sigma \in \ker(\operatorname{div}\operatorname{div} : \hat{H}_{\mathrm{dd}})$,

$$\operatorname{div}\operatorname{sym}\operatorname{curl}\tilde{R}_{\mathrm{cc}}\sigma = \operatorname{div}\sigma, \qquad \|\tilde{R}_{\mathrm{cc}}\sigma\|_{H^1} \lesssim \|\operatorname{div}\sigma\|_{\tilde{\mathring{H}}^{-1}}.$$

Proof. Consider a $\sigma \in \hat{H}_{\mathrm{dd}}$ with $\operatorname{div}\operatorname{div}\sigma = 0$. Then, since $\operatorname{div}\sigma \in \tilde{\mathring{H}}^{-1} \otimes \mathbb{V}$ has vanishing divergence, $u = T_{\mathrm{c}}\operatorname{div}\sigma$ is in $L_2 \otimes \mathbb{V}$ and satisfies $\operatorname{curl} u = \operatorname{div}\sigma$ by (43c). Next, we claim that $(u, b) = 0$ for any $b \in \mathbb{V}$. To see this, first note that the distribution $\operatorname{div}\sigma$ satisfies

$$(\operatorname{div}\sigma)(b \times x) = \sigma(\operatorname{grad}(b \times x)) = \sigma(\operatorname{mskw} b) = 0$$

due to the symmetry of σ. Relating to u,

$$0 = (\operatorname{div}\sigma)(b \times x) = (\operatorname{curl} u)(b \times x) = (u, \operatorname{curl}(b \times x)) = 2(u, b).$$

Hence we may apply T_{d} to each component of $\frac{1}{2}u$ and use (43d) to get a $\tau \in \mathring{H}^1 \otimes \mathbb{M}$ such that $\operatorname{div}\tau = \frac{1}{2}u$, which implies

$$\operatorname{div}\sigma = \operatorname{curl} u = \frac{1}{2}\operatorname{curl}\operatorname{div}\tau = \frac{1}{2}\operatorname{curl}\operatorname{div}\operatorname{dev}\tau = \operatorname{div}\operatorname{sym}\operatorname{curl}\top\operatorname{dev}\tau.$$

Here we have used (30) and the fact that $\operatorname{curl}\operatorname{div}$ vanishes on matrix fields that are scalar multiples of the identity. Denoting the map $\sigma \mapsto \top\operatorname{dev}\tau$ by $\tilde{R}_{\mathrm{cc}}$, the continuity of T_{c} and T_{d} finishes the proof. $\square$

Theorem 3.7. *There exist three continuous linear operators*

$$\mathring{S}^{(0)}_{\mathrm{dd}} : \hat{H}_{\mathrm{dd}} \to \mathring{H}^1 \otimes \mathbb{S}, \qquad \mathring{S}^{(1)}_{\mathrm{dd}} : \hat{H}_{\mathrm{dd}} \to \mathring{H}^1 \otimes \mathbb{T}, \qquad \mathring{S}^{(2)}_{\mathrm{dd}} : \hat{H}_{\mathrm{dd}} \to \mathring{H}^1 \otimes \mathbb{S},$$

such that any $\sigma \in \hat{H}_{\mathrm{dd}}$ *can be decomposed into*

$$\sigma = \mathring{S}^{(0)}_{\mathrm{dd}}\,\sigma + \operatorname{sym}\operatorname{curl}\mathring{S}^{(1)}_{\mathrm{dd}}\,\sigma + \operatorname{inc}\mathring{S}^{(2)}_{\mathrm{dd}}\,\sigma. \tag{48}$$

Consequently, $\hat{H}_{\mathrm{dd}} = \tilde{\mathring{H}}_{\mathrm{dd}}$.

Proof. Let $\sigma \in \hat{H}_{\mathrm{dd}}$ and $\mathring{S}^{(0)}_{\mathrm{dd}}\,\sigma := D_{\mathrm{dd}}\operatorname{div}\operatorname{div}\sigma$. Note that $\operatorname{div}\operatorname{div}\sigma$ is in $\tilde{\mathring{H}}^{-1}_{\mathcal{P}_1}$, the domain of D_{dd}, because the hessian of p is zero for any $p \in \mathcal{P}_1$ and

$$(\operatorname{div}\operatorname{div}\sigma)(p) = \sigma(\operatorname{hess} p) = 0.$$

By Lemma 3.5,

$$\operatorname{div}\operatorname{div}(\sigma - \mathring{S}^{(0)}_{\mathrm{dd}}\,\sigma) = 0.$$

Next, set $\mathring{S}^{(1)}_{\mathrm{dd}}\,\sigma := \tilde{R}_{\mathrm{cc}}(\sigma - \mathring{S}^{(0)}_{\mathrm{dd}}\,\sigma)$ in $\mathring{H}^1 \otimes \mathbb{T}$. By Lemma 3.6,

$$\operatorname{div}\left(\sigma - \mathring{S}^{(0)}_{\mathrm{dd}}\,\sigma - \operatorname{sym}\operatorname{curl}\mathring{S}^{(1)}_{\mathrm{dd}}\,\sigma\right) = 0.$$

By Lemma 3.1, setting $\mathring{S}^{(2)}_{\mathrm{dd}}\,\sigma := D_{\mathrm{cc}}(\sigma - \mathring{S}^{(0)}_{\mathrm{dd}}\,\sigma - \operatorname{sym}\operatorname{curl}\mathring{S}^{(1)}_{\mathrm{dd}}\,\sigma)$, we find that

$$\sigma - \mathring{S}^{(0)}_{\mathrm{dd}}\,\sigma - \operatorname{sym}\operatorname{curl}\mathring{S}^{(1)}_{\mathrm{dd}}\,\sigma = \operatorname{inc}\mathring{S}^{(2)}_{\mathrm{dd}}\,\sigma,$$

thus completing the proof of (48).

To conclude, it suffices to prove that $\hat{H}_{\mathrm{dd}} \subseteq \tilde{\mathring{H}}_{\mathrm{dd}}$, due to (42). Decompose any $\sigma \in \hat{H}_{\mathrm{dd}}$ into $\sigma = \mathring{S}^{(0)}_{\mathrm{dd}}\,\sigma + \operatorname{sym}\operatorname{curl}\mathring{S}^{(1)}_{\mathrm{dd}}\,\sigma + \operatorname{inc}\mathring{S}^{(2)}_{\mathrm{dd}}\,\sigma$. By the density of $\mathcal{D}(\Omega)$ in $\mathring{H}^1$, there are $\sigma_m \in \mathcal{D}(\Omega)\otimes\mathbb{S}$, $\tau_m \in \mathcal{D}(\Omega)\otimes\mathbb{T}$, and $g_m \in \mathcal{D}(\Omega)\otimes\mathbb{S}$ such that

$$\|\sigma_m - \mathring{S}^{(0)}_{\mathrm{dd}}\,\sigma\|_{H^1} \to 0, \qquad \|\tau_m - \mathring{S}^{(1)}_{\mathrm{dd}}\,\sigma\|_{H^1} \to 0, \qquad \|g_m - \mathring{S}^{(2)}_{\mathrm{dd}}\,\sigma\|_{H^1} \to 0,$$

as $m \to \infty$. Hence, by (18), $\sigma_m + \operatorname{sym}\operatorname{curl}\tau_m + \operatorname{inc} g_m \in \mathcal{D}(\Omega)\otimes\mathbb{S}$ converges to σ in $\|\cdot\|_{\hat{\mathring{H}}_{\mathrm{dd}}}$ norm, thus proving that $\sigma \in \tilde{\mathring{H}}_{\mathrm{dd}}$. □

3.3. **Regular decomposition of $\tilde{\mathring{H}}_{\mathrm{cd}}$.** Next, we turn to constructing a regular decomposition of $\tilde{\mathring{H}}_{\mathrm{cd}}$. (The case of $\tilde{\mathring{H}}_{\mathrm{cd}\top}$ obviously follows from that of $\tilde{\mathring{H}}_{\mathrm{cd}}$.) Unlike the three-term decompositions of $\tilde{\mathring{H}}_{\mathrm{cc}}$ and $\tilde{\mathring{H}}_{\mathrm{dd}}$ cases, now we are only able to construct a decomposition with four terms. We begin with a preparatory lemma.

Lemma 3.8. *Let* $K = \ker(\operatorname{curl}\operatorname{div} : \hat{H}_{\mathrm{cd}})$. *There are linear maps* $\hat{R}_{\mathrm{gc}} : K \to \mathring{H}^1 \otimes \mathbb{S}$, $\hat{D}_{\mathrm{gc}} : K \to \mathring{H}^1 \otimes \mathbb{V}$, *and* $\hat{U}_{\mathrm{gc}} : K \to \mathring{H}^1 \otimes \mathbb{V}$, *such that any* $\tau \in \hat{H}_{\mathrm{cd}}$ *with* $\operatorname{curl}\operatorname{div}\tau = 0$ *can be decomposed into*

$$\tau = \operatorname{curl}\hat{R}_{\mathrm{gc}}\tau + \operatorname{curl}\operatorname{def}\hat{U}_{\mathrm{gc}}\,\tau + \top\operatorname{dev}\operatorname{grad}\hat{D}_{\mathrm{gc}}\,\tau \tag{49}$$

and the following continuity bound holds:

$$\|\hat{R}_{\mathrm{gc}}\tau\|_{H^1} + \|\,\hat{D}_{\mathrm{gc}}\,\tau\|_{H^1} + \|\,\hat{U}_{\mathrm{gc}}\,\tau\|_{H^1} \lesssim \|\tau\|_{\hat{\mathring{H}}_{\mathrm{cd}}}. \tag{50}$$

If in addition, τ *is in* $L_2 \otimes \mathbb{T}$, *then* $\hat{U}_{\mathrm{gc}}$ *can be taken to be zero provided* (50) *is replaced by*

$$\|\hat{R}_{\mathrm{gc}}\tau\|_{H^1} \lesssim \|\tau\|_{\hat{\mathring{H}}_{\mathrm{cd}}}, \qquad \|\,\hat{D}_{\mathrm{gc}}\,\tau\|_{H^1} \lesssim \|\tau\|_{L_2} + \|\tau\|_{\hat{\mathring{H}}_{\mathrm{cd}}}.$$

Proof. Given any $\tau \in \hat{H}_{\mathrm{cd}}$ with $\operatorname{curl}\operatorname{div}\tau = 0$, put $w = T_{\mathrm{g}}\operatorname{div}\tau$. Then by (43b),

$$\operatorname{grad} w = \operatorname{div}\tau, \qquad \|w\|_{L_2} \lesssim \|\operatorname{div}\tau\|_{\mathring{H}^{-1}}. \tag{51}$$

Since $\operatorname{tr}(\top\tau) = 0$, we know that $S\top\tau = \tau$, so

$$\operatorname{skw}\operatorname{curl}\top\tau = \frac{1}{2}\operatorname{mskw}\operatorname{div} S\top\tau = \frac{1}{2}\operatorname{mskw}\operatorname{div}\tau = \frac{1}{2}\operatorname{mskw}\operatorname{grad} w = -\frac{1}{2}\operatorname{curl}(w\mathbb{I}), \tag{52}$$

where we have used (26d) and (26a).

Let $\sigma = \operatorname{sym}\operatorname{curl}\top\tau$. By the identity (30) of Lemma 2.1, $\operatorname{div}\sigma = \operatorname{curl}\operatorname{div}\tau = 0$, so applying Lemma 3.2, $g = \tilde{R}_{\mathrm{gg}}\sigma$ is in $\mathring{H}^1 \otimes \mathbb{S}$ and satisfies

$$\sigma = \operatorname{inc} g, \qquad \|g\|_{H^1} \lesssim \|\operatorname{sym}\operatorname{curl}\top\tau\|_{\mathring{H}^{-1}} \tag{53}$$

Combined with (52), we have the twin identities

$$\operatorname{sym}\operatorname{curl}\top\tau = \operatorname{curl}\top\operatorname{curl} g,$$

$$\operatorname{skw}\operatorname{curl}\top\tau = -\frac{1}{2}\operatorname{curl}(w\mathbb{I}).$$

Adding these equations, we find that $\operatorname{curl}(\top\tau - \top\operatorname{curl} g + \frac{1}{2}w\mathbb{I}) = 0$. Hence, applying T_{g} to each of the row vectors of $\top\tau - \top\operatorname{curl} g + \frac{1}{2}w\mathbb{I}$ and using (43b), we obtain a $q \in L_2 \otimes \mathbb{V}$ such that

$$\operatorname{grad} q = \top\tau - \top\operatorname{curl} g + \frac{1}{2}w\mathbb{I}, \tag{54a}$$

$$\|q\|_{L_2} \lesssim \|\tau - \operatorname{curl} g + w\|_{\mathring{H}^{-1}}. \tag{54b}$$

In fact, $q|_\Omega$ is in $\mathring{H}(\mathrm{div})$. To see this, take traces on both sides of (54a). Recall that $\operatorname{tr}\tau = 0$. Also, $\operatorname{tr}(\operatorname{curl} g) = 0$ by (26f). Hence we conclude that $\frac{3}{2}w = \operatorname{div} q$, an identity that holds in all $\mathbb{R}^3$ with q and w supported only on $\bar{\Omega}$. Since $w \in L_2$, this in particular shows that $q|_\Omega \in \mathring{H}(\mathrm{div})$, and the estimate

$$\|q\|_{H(\mathrm{div})} \lesssim \|\tau\|_{\hat{H}^{-1}} + \|g\|_{H^1} + \|w\|_{L_2} \lesssim \|\tau\|_{\hat{\mathring{H}}_{\mathrm{cd}}} \tag{55}$$

follows from the estimates of (54b), (53), and (51).

Taking the deviatoric part of both sides of (54a) and noting that $\tau = \operatorname{dev}\tau$, we obtain a preliminary two-term decomposition of τ,

$$\tau = \operatorname{curl} g + \top \operatorname{dev} \operatorname{grad} q. \tag{56}$$

However, here q is not in $\mathring{H}^1 \otimes \mathbb{V}$, in general. To improve this to the needed result, we use $r = T_{\mathrm{d}} \operatorname{div} q$, which has the same divergence as q, but is in $\mathring{H}^1 \otimes \mathbb{V}$:

$$\operatorname{div} r = \operatorname{div} q, \qquad \|r\|_{H^1} \lesssim \|\operatorname{div} q\|_{L_2} \lesssim \|\tau\|_{\hat{\mathring{H}}_{\mathrm{cd}}},$$

by (55). Since $\operatorname{div}(q - r) = 0$, putting $u = \frac{1}{2}T_{\mathrm{c}}(q - r)$ in $\mathring{H}^1 \otimes \mathbb{V}$, by (43c),

$$\frac{1}{2}\operatorname{curl} u = q - r, \qquad \|u\|_{H^1} \lesssim \|q - r\|_{L_2} \lesssim \|\tau\|_{\hat{\mathring{H}}_{\mathrm{cd}}}.$$

Hence

$$\operatorname{dev}\operatorname{grad} q = \operatorname{dev}\operatorname{grad} r + \frac{1}{2}\operatorname{dev}\operatorname{grad}\operatorname{curl} u = \operatorname{dev}\operatorname{grad} r + \operatorname{curl}\operatorname{def} u.$$

Substituting this into (56) and setting $\hat{R}_{\mathrm{gc}}\tau = g$, $\hat{D}_{\mathrm{gc}}\,\tau = r$, and $\hat{U}_{\mathrm{gc}}\,\tau = u$, we see that (49) and (50) follow.

To prove the remaining statement, suppose $\tau \in L_2 \otimes \mathbb{T} \cap \hat{H}_{\mathrm{cd}}$. Then, due to the higher regularity of τ, observe that q in (54a) is in $\mathring{H}^1 \otimes \mathbb{V}$ and in place of (54b), we have

$$\|q\|_{H^1} \lesssim \|\tau - \operatorname{curl} g + w\|_{L_2} \lesssim \|\tau\|_{L_2} + \|g\|_{H^1} + \|w\|_{L_2}.$$

which can be used in place of (55). There is no longer a need to produce the r above, and we may set $\hat{D}_{\mathrm{gc}}\,\tau = q \in \mathring{H}^1 \otimes \mathbb{V}$. The decomposition (56) then concludes the proof. □

Lemma 3.9. *There is a linear map* $D_{\mathrm{cd}} : \ker(\operatorname{div} : \tilde{\mathring{H}}^s_{\mathcal{ND}}(\mathrm{div})) \to \tilde{\mathring{H}}^{s+2} \otimes \mathbb{T}$ *such that*

$$\operatorname{curl}\operatorname{div} D_{\mathrm{cd}} v = v, \qquad \|D_{\mathrm{cd}} v\|_{\tilde{\mathring{H}}^{s+2}} \lesssim \|v\|_{\tilde{\mathring{H}}^s},$$

for any $s \in \mathbb{R}$ *and* v *in* $\tilde{\mathring{H}}^s_{\mathcal{ND}}(\mathrm{div})$ *with zero divergence.*

Proof. Since $\operatorname{div} v = 0$, by (43c), $u = T_{\mathrm{c}} v$ is in $\tilde{\mathring{H}}^s \otimes \mathbb{V}$ and satisfies $\operatorname{curl} u = v$ in all $\mathbb{R}^3$. For any constant vector $b \in \mathbb{V}$, the action of the distribution u on b satisfies

$$2\,u(b) = u(\operatorname{curl}(b \times x)) = (\operatorname{curl} u)(b \times x) = v(b \times x) = 0$$

since $v(r) = 0$ for any $r \in \mathcal{ND}$. Hence, applying T_{d} to each component of u and using (43d), we obtain a $\tau \in \tilde{\mathring{H}}^{s+2} \otimes \mathbb{M}$ such that $\operatorname{div}\tau = u$, i.e.,

$$v = \operatorname{curl}\operatorname{div}\tau = \operatorname{curl}\operatorname{div}\operatorname{dev}\tau,$$

since $\operatorname{curl}\operatorname{div}(\frac{1}{3}(\operatorname{tr}\tau)\mathbb{1}) = 0$. Denoting the map $v \mapsto \operatorname{dev}\tau$ by D_{cd}, the proof is finished by the continuity of T_{c} and T_{d}. □

Theorem 3.10. *There exist four continuous linear operators*

$\mathring{\hat{S}}^{(0)}_{\rm cd} : \hat{H}_{\rm cd} \to \mathring{H}^1 \otimes \mathbb{T}, \quad \mathring{\hat{S}}^{(1)}_{\rm cd} : \hat{H}_{\rm cd} \to \mathring{H}^1 \otimes \mathbb{S}, \quad \mathring{\hat{S}}^{(2)}_{\rm cd} : \hat{H}_{\rm cd} \to \mathring{H}^1 \otimes \mathbb{V}, \quad \mathring{\hat{S}}^{(3)}_{\rm cd} : \hat{H}_{\rm cd} \to \mathring{H}^1 \otimes \mathbb{V},$

such that any $\tau \in \hat{H}_{\rm cd}$ *can be decomposed into*

$$\tau = \mathring{\hat{S}}^{(0)}_{\rm cd}\,\tau + \operatorname{curl} \mathring{\hat{S}}^{(1)}_{\rm cd}\,\tau + \top \operatorname{dev}\operatorname{grad} \mathring{\hat{S}}^{(2)}_{\rm cd}\,\tau + \operatorname{curl}\operatorname{def} \mathring{\hat{S}}^{(3)}_{\rm cd}\,\tau. \tag{57}$$

It then follows that $\hat{H}_{\rm cd} = \mathring{\tilde{H}}_{\rm cd}$.

Proof. Let $\tau \in \hat{H}_{\rm cd}$ and put $q = \operatorname{curl}\operatorname{div}\tau$. Obviously $\operatorname{div} q = 0$ and $q \in \mathring{\tilde{H}}^{-1}(\operatorname{div})$. Moreover, for any $a, b \in \mathbb{V}$ and $r = a + b \times x \in \mathcal{ND}$, since $\operatorname{curl} r = 2b$ is constant, its gradient vanishes, and

$$q(r) = (\operatorname{curl}\operatorname{div}\tau)(r) = \tau(\operatorname{grad}\operatorname{curl} r) = 0.$$

Thus q is in $\mathring{\tilde{H}}^{-1}_{\mathcal{ND}}(\operatorname{div})$ and we apply $D_{\rm cd}$ to it. Put $\mathring{\hat{S}}^{(0)}_{\rm cd}\,\tau := D_{\rm cd}\operatorname{curl}\operatorname{div}\tau$. By Lemma 3.9 with $s = -1$, we find that $\mathring{\hat{S}}^{(0)}_{\rm cd}\,\tau \in \mathring{H}^1 \otimes \mathbb{T}$ and

$$\operatorname{curl}\operatorname{div}(\tau - \mathring{\hat{S}}^{(0)}_{\rm cd}\,\tau) = 0.$$

Hence we may apply Lemma 3.8 to get

$$\tau - \mathring{\hat{S}}^{(0)}_{\rm cd}\,\tau = (\operatorname{curl}\hat{R}_{\rm gc} + \operatorname{curl}\operatorname{def}\hat{U}_{\rm gc} + \top\operatorname{dev}\operatorname{grad}\hat{D}_{\rm gc})(\tau - \mathring{\hat{S}}^{(0)}_{\rm cd}\,\tau).$$

The decomposition (57) now follows after setting $\mathring{\hat{S}}^{(1)}_{\rm cd}\,\tau = \hat{R}_{\rm gc}(\tau - \mathring{\hat{S}}^{(0)}_{\rm cd}\,\tau)$, $\mathring{\hat{S}}^{(2)}_{\rm cd}\,\tau = \hat{D}_{\rm gc}(\tau - \mathring{\hat{S}}^{(0)}_{\rm cd}\,\tau)$ and $\mathring{\hat{S}}^{(3)}_{\rm cd}\,\tau = \hat{U}_{\rm gc}(\tau - \mathring{\hat{S}}^{(0)}_{\rm cd}\,\tau)$.

To prove that $\hat{H}_{\rm cd} = \mathring{\tilde{H}}_{\rm cd}$, let $\tau \in \hat{H}_{\rm cd}$ be decomposed as in (57). By the density of $\mathcal{D}(\Omega)$ in $\mathring{H}^1$, there are $\tau_m \in \mathcal{D}(\Omega) \otimes \mathbb{T}$, $g_m \in \mathcal{D}(\Omega) \otimes \mathbb{S}$, $q_m \in \mathcal{D}(\Omega) \otimes \mathbb{V}$, and $u_m \in \mathcal{D}(\Omega) \otimes \mathbb{V}$, such that

$$\|\tau_m - \mathring{\hat{S}}^{(0)}_{\rm cd}\,\tau\|_{H^1} \to 0, \quad \|g_m - \mathring{\hat{S}}^{(1)}_{\rm cd}\,\tau\|_{H^1} \to 0, \quad \|q_m - \mathring{\hat{S}}^{(2)}_{\rm cd}\,\tau\|_{H^1} \to 0, \quad \|u_m - \mathring{\hat{S}}^{(2)}_{\rm dd}\,\tau\|_{H^1} \to 0,$$

as $m \to \infty$. By (31),

$$\begin{aligned}
&\|\tau_m + \operatorname{curl} g_m + \top\operatorname{dev}\operatorname{grad} q_m + \operatorname{curl}\operatorname{def} u_m \; - \; \tau\|_{\mathring{\tilde{H}}_{\rm cd}} \\
&\quad = \|\tau_m - \tau\|_{\mathring{\tilde{H}}_{\rm cd}} + \|\operatorname{curl}(g_m - \mathring{\hat{S}}^{(1)}_{\rm cd}\,\tau)\|_{L_2} + \|\top\operatorname{dev}\operatorname{grad}(q_m - \mathring{\hat{S}}^{(2)}_{\rm cd}\,\tau)\|_{L_2} \\
&\quad + \|\operatorname{curl}\operatorname{def}(u_m - \mathring{\hat{S}}^{(3)}_{\rm cd}\,\tau)\|_{L_2} \to 0,
\end{aligned}$$

which converges to zero as $m \to \infty$ in view of (18). Thus $\hat{H}_{\rm cd} \subseteq \mathring{\tilde{H}}_{\rm cd}$ and the proof is complete due to (42). $\square$

In view of these results, we shall no longer distinguish between $\hat{H}_{\rm cc}$ and $\mathring{\tilde{H}}_{\rm cc}$, $\hat{H}_{\rm dd}$ and $\mathring{\tilde{H}}_{\rm dd}$, nor $\hat{H}_{\rm cd}$ and $\mathring{\tilde{H}}_{\rm cd}$.

3.4. **Continuous right inverses.** Let us now complete the discussion of (41). Several right inverse operators in (41) were already given in previous lemmas. The right inverses in the top row of (41) are the same operators as in (43b)–(43d). For example, $T_{\rm c}$ is a right inverse of $\operatorname{curl} : \mathring{H}(\operatorname{curl}) \to \mathring{H}(\operatorname{div})$ in the sense that $T_{\rm c} : \ker(\operatorname{div} : \mathring{H}(\operatorname{div})) \to \mathring{H}(\operatorname{curl})$ is continuous and $\operatorname{curl} \circ T_{\rm c}$ equals the identity on $\ker(\operatorname{div} : \mathring{H}(\operatorname{div}))$, which is just a restatement of (43c) with $s = 0$. After construction of the remaining needed right inverses, Theorem 3.21 below gathers everything together.

Lemma 3.11. *There are linear maps* $\tilde{R}_{\rm gc} : \ker(\operatorname{div} : \mathring{\tilde{H}}_{\rm cd}) \to L_2 \otimes \mathbb{S} \subset \mathring{\tilde{H}}_{\rm cc}$ *and* $\tilde{D}_{\rm gc} : \ker(\operatorname{div} : \mathring{\tilde{H}}_{\rm cd}) \to \mathring{H}^1 \otimes \mathbb{V} \subset \mathring{H}(\operatorname{curl})$ *such that for all* $\tau \in \ker(\operatorname{div} : \mathring{\tilde{H}}_{\rm cd})$,

$$\tau = \operatorname{curl}(\tilde{R}_{\rm gc}\tau + \operatorname{def} \tilde{D}_{\rm gc}\, \tau), \qquad \|\tilde{R}_{\rm gc}\tau\|_{L_2} + \|\, \tilde{D}_{\rm gc}\, \tau\|_{H^1} \lesssim \|\tau\|_{\mathring{\tilde{H}}^{-1}}.$$

Proof. Applying $T_{\rm c}$ to the divergence-free row vectors of τ, we find a $\gamma \in L_2 \otimes \mathbb{M}$ satisfying $\operatorname{curl}\gamma = \tau$ per (43c). Put $g = \operatorname{sym}\gamma$ and $v = \operatorname{vskw}\gamma$. Then, by (26e),

$$\begin{aligned} \tau &= \operatorname{curl}(\operatorname{sym}\gamma + \operatorname{skw}\gamma) = \operatorname{curl} g - \operatorname{curl} \operatorname{mskw} v \\ &= \operatorname{curl} g - S \operatorname{grad} v \\ &= \operatorname{curl} g - \top \operatorname{grad} v + (\operatorname{div} v)\mathbb{I}. \end{aligned}$$

Since $\mathring{\tilde{H}}_{\rm cd}$ consists of trace-free matrix fields and since trace of $\operatorname{curl} g$ vanishes by (26f), taking the trace of the above expression, we find that

$$0 = \operatorname{tr}\tau = 2 \operatorname{div} v.$$

Therefore, by (43c), $u = -\frac{1}{2} T_{\rm c} v \in \mathring{H}^1 \otimes \mathbb{V}$, satisfies $v = -\frac{1}{2}\operatorname{curl} u$, so

$$\begin{aligned} \tau &= \operatorname{curl} g + \frac{1}{2} \top \operatorname{grad} \operatorname{curl} u \\ &= \operatorname{curl}(g + \operatorname{def} u) \end{aligned}$$

by (31) of Lemma 2.1. The linear maps $\tau \mapsto g$ and $\tau \mapsto u$ are the needed $\tilde{R}_{\rm gc}$ and $\tilde{D}_{\rm gc}$. □

Lemma 3.12. *There is a linear map* $R_{\rm gc} : \ker(\operatorname{div} : \mathring{\tilde{H}}_{\rm cd}) \to L_2 \otimes \mathbb{S} \subset \mathring{\tilde{H}}_{\rm cc}$ *such that for all* $\tau \in \ker(\operatorname{div} : \mathring{\tilde{H}}_{\rm cd})$,

$$\tau = \operatorname{curl} R_{\rm gc}\tau, \qquad \|R_{\rm gc}\tau\|_{L_2} \lesssim \|\tau\|_{\mathring{\tilde{H}}^{-1}}.$$

Proof. Using the operators of Lemma 3.11, define $R_{\rm gc}\tau = \tilde{R}_{\rm gc}\tau + \operatorname{def} \tilde{D}_{\rm gc}\, \tau$. Then the result follows immediately from Lemmas 3.11 and (18). □

Lemma 3.13. *There is a linear map* $D_{\rm gc} : \ker(\operatorname{div} : \mathring{\tilde{H}}_{\rm cd}) \cap \ker(\operatorname{sym} \operatorname{curl} \top : \mathring{\tilde{H}}_{\rm cd}) \to \mathring{H}^1 \otimes \mathbb{V} \subset \mathring{H}(\operatorname{curl})$ *such that for all* $\tau \in \mathring{\tilde{H}}_{\rm cd}$ *with* $\operatorname{div}\tau = 0$ *and* $\operatorname{sym}\operatorname{curl}\top\tau = 0$,

$$\tau = \operatorname{curl} \operatorname{def} D_{\rm gc}\, \tau, \qquad \|\, D_{\rm gc}\, \tau\|_{H^1} \lesssim \|\tau\|_{\mathring{\tilde{H}}^{-1}}$$

Proof. Given any $\tau \in \mathring{\tilde{H}}_{\rm cd}$ with $\operatorname{div}\tau = 0$, by Lemma 3.11, $\tau = \operatorname{curl}(\tilde{R}_{\rm gc}\tau + \operatorname{def} \tilde{D}_{\rm gc}\, \tau)$. When $\operatorname{sym}\operatorname{curl}\top\tau$ also vanishes, this implies that

$$0 = \operatorname{sym}\operatorname{curl}\top\tau = \operatorname{sym}\operatorname{curl}\top\operatorname{curl}(\tilde{R}_{\rm gc}\tau + \operatorname{def}\tilde{D}_{\rm gc}\, \tau) = \operatorname{inc}(\tilde{R}_{\rm gc}\tau).$$

Applying Lemma 3.2 with $g = \tilde{R}_{\rm gc}\tau$, $\operatorname{curl}\operatorname{def}\tilde{R}_{\rm gg} g = \operatorname{curl} g$, which in turn implies that

$$\tau = \operatorname{curl}(\tilde{R}_{\rm gc}\tau + \operatorname{def}\tilde{D}_{\rm gc}\, \tau) = \operatorname{curl}\operatorname{def}(\tilde{R}_{\rm gg}\tilde{R}_{\rm gc}\tau + \tilde{D}_{\rm gc}\, \tau).$$

Hence the result follows by setting $D_{\rm gc} = \tilde{R}_{\rm gg}\tilde{R}_{\rm gc} + \tilde{D}_{\rm gc}$. □

Lemma 3.14. *There is a linear map* $D_{\rm gd} : \ker(\operatorname{curl} : \mathring{\tilde{H}}^{-1}_{\mathcal{RT}}(\operatorname{curl})) \to \mathring{H}^1 \otimes \mathbb{V}$ *such that for all* $v \in \ker(\operatorname{curl} : \mathring{\tilde{H}}^{-1}_{\mathcal{RT}}(\operatorname{curl}))$

$$v = \frac{1}{3} \operatorname{grad} \operatorname{div} D_{\rm gd} v, \qquad \|D_{\rm gd} v\|_{H^1} \lesssim \|v\|_{\mathring{\tilde{H}}^{-1}}. \tag{58}$$

Proof. Let $v \in \mathring{\tilde{H}}^{-1}_{\mathcal{RT}}(\mathrm{curl})$ have zero curl. Then $w = T_{\mathrm{g}} v$ is in $L_2(\mathbb{R}^3) \otimes \mathbb{V}$, supported on $\bar{\Omega}$, and satisfies $v = \mathrm{grad}\, w$ in all $\mathbb{R}^3$ due to (43b). Since $v(r) = 0$ for all $r \in \mathcal{RT}$, choosing $r = bx$ for any constant b,

$$0 = (\mathrm{grad}\, w)(r) = (w, \mathrm{div}\, r) = 3(w, b).$$

Hence we may apply T_{d} to each component of w and use (43d) to get a $q \in \mathring{H}^1 \otimes \mathbb{V}$ satisfying $w = \frac{1}{3}\,\mathrm{grad}\,\mathrm{div}\, q$. The linear map $v \mapsto w$ is the required D_{gd}. □

Lemma 3.15. *There is a linear map* $R_{\mathrm{gg}} : \ker(\mathrm{inc} : \mathring{\tilde{H}}_{\mathrm{cc}}) \to \mathring{H}(\mathrm{curl})$ *such that for any* $g \in \ker(\mathrm{inc} : \mathring{\tilde{H}}_{\mathrm{cc}})$,

$$\mathrm{def}\, R_{\mathrm{gg}} g = g, \qquad \|R_{\mathrm{gg}} g\|_{H(\mathrm{curl})} \lesssim \|g\|_{\mathring{\tilde{H}}_{\mathrm{cc}}}.$$

Proof. By Lemma 3.2, $u = \tilde{R}_{\mathrm{gg}} g$ satisfies $\mathrm{curl}(g - \mathrm{def}\, u) = 0$. Hence applying T_{g} to each row vector of $g - \mathrm{def}\, u$, we obtain a $v \in L_2 \otimes \mathbb{V}$ such that $g - \mathrm{def}\, u = \mathrm{grad}\, v$. The symmetry of the left hand side implies that

$$0 = \mathrm{skw}(g - \mathrm{def}\, u) = \mathrm{skw}\,\mathrm{grad}\, v = \frac{1}{2}\,\mathrm{mskw}\,\mathrm{curl}\, v$$

by (26c). Hence $\mathrm{curl}\, v = 0$ on all $\mathbb{R}^3$, so the vector field $v|_\Omega$ is in $\mathring{H}(\mathrm{curl})$. We have thus shown that $g = \mathrm{def}(u + v)$. Letting the map $g \mapsto u + v$ be denoted by R_{gg}, the norm bound follows from the continuity of T_{g} and $\tilde{R}_{\mathrm{gg}}$. □

Lemma 3.16. *There is a linear map* $R_{\mathrm{gd}} : \mathring{\tilde{H}}^{-1}_{\mathcal{RT}}(\mathrm{curl}) \to L_2 \otimes \mathbb{T} \subset \mathring{\tilde{H}}_{\mathrm{cd}}$ *such that for all* $v \in \mathring{\tilde{H}}^{-1}_{\mathcal{RT}}(\mathrm{curl})$,

$$v = \mathrm{div}\, R_{\mathrm{gd}} v, \qquad \|R_{\mathrm{gd}} v\|_{L_2} \lesssim \|v\|_{\mathring{\tilde{H}}^{-1}}. \tag{59}$$

Proof. Let $v \in \mathcal{D}_{\mathcal{RT}}$. Since every component v_i of v has zero mean on Ω, we may apply T_{d} to each and use (43d) to get a $\tau \in L_2(\mathbb{R}^3) \otimes \mathbb{M}$, supported on $\bar{\Omega}$, satisfying $v = \mathrm{div}\, \tau$ on all $\mathbb{R}^3$, which in particular, implies that τ is in $\mathring{H}(\mathrm{div})$. Hence we may integrate by parts to conclude that

$$(\tau, \mathrm{grad}\, r) = (\mathrm{div}\, \tau, r) = (v, r) = 0$$

for all $r \in \mathcal{RT}$. Choosing $r = bx$ for any $b \in \mathbb{R}$ and noting that $\mathrm{grad}(bx) = b\mathbb{i}$, we find that $(\tau, \mathbb{i}) = 0$. Hence $t = \mathrm{tr}(\tau) \in L_2$ satisfies

$$0 = (\tau, \mathbb{i}) = (\mathrm{dev}\, \tau, \mathbb{i}) + \frac{1}{3}(t\mathbb{i}, \mathbb{i}) = (t, 1).$$

Now, by (43d), $q = T_{\mathrm{d}} t$ satisfies $\mathrm{div}\, q = t$, so

$$\begin{aligned} v = \mathrm{div}\, \tau &= \mathrm{div}\,\mathrm{dev}\, \tau + \frac{1}{3}\,\mathrm{div}(\mathrm{tr}\, \tau)\, \mathbb{i} = \mathrm{div}\,\mathrm{dev}\, \tau + \frac{1}{3}\,\mathrm{grad}(t) \\ &= \mathrm{div}(\mathrm{dev}\, \tau) + \frac{1}{3}\,\mathrm{grad}(\mathrm{div}\, q) \\ &= \mathrm{div}(\mathrm{dev}\, \tau) + \frac{1}{2}\,\mathrm{div}(\top\, \mathrm{dev}\,\mathrm{grad}\, q) \end{aligned}$$

by (32) of Lemma 2.1. Denoting the linear maps $v \mapsto \mathrm{dev}\, \tau + \frac{1}{2} \top\, \mathrm{dev}\,\mathrm{grad}\, q$ by R_{gd}, the norm estimate in (59) follows for $v \in \mathcal{D}_{\mathcal{RT}}$. The proof is now finished by the density result of Lemma 2.2. □

Lemma 3.17. *There is a linear map* $R_{\mathrm{gc}\top} : \ker(\operatorname{sym}\operatorname{curl} : \mathring{\tilde{H}}_{\mathrm{cd}\top}) \to \mathring{H}(\operatorname{div})$ *such that for all* $\tau \in \ker(\operatorname{sym}\operatorname{curl} : \mathring{\tilde{H}}_{\mathrm{cd}\top})$,

$$\frac{1}{2}\operatorname{dev}\operatorname{grad} R_{\mathrm{gc}\top}\tau = \tau, \qquad \|R_{\mathrm{gc}\top}\tau\|_{H(\operatorname{div})} \lesssim \|\tau\|_{\mathring{\tilde{H}}_{\mathrm{cd}}}.$$

Proof. Let $\tau \in \ker(\operatorname{sym}\operatorname{curl} : \mathring{\tilde{H}}_{\mathrm{cd}\top})$. Then $\operatorname{curl}\operatorname{div} \top\tau = \operatorname{div}\operatorname{sym}\operatorname{curl}\tau = 0$, so by (43b), $w = T_{\mathrm{g}}\operatorname{div}(\top\tau)$ in $L_2(\mathbb{R}^3)$, supported on $\bar{\Omega}$, satisfies $\operatorname{grad} w = \operatorname{div}\top\tau$.

Next, recalling that $\operatorname{tr}\tau = 0$, note that

$$\begin{aligned} 2\operatorname{skw}\operatorname{curl}\tau &= \operatorname{mskw}\operatorname{div} S\tau && \text{by (26d)} \\ &= \operatorname{mskw}\operatorname{div}\top\tau = \operatorname{mskw}\operatorname{grad} w \\ &= -\operatorname{curl}(w\mathbb{I}), && \text{by (26b)}. \end{aligned}$$

Hence $\operatorname{curl}\tau = \operatorname{sym}\operatorname{curl}\tau + \operatorname{skw}\operatorname{curl}\tau = -\frac{1}{2}\operatorname{curl}(w\mathbb{I})$. Applying T_{g} to each row vector of $\tau + \frac{1}{2}w\mathbb{I}$, (43b) we obtain a $q \in L_2 \otimes \mathbb{V}$ satisfying

$$\tau + \frac{1}{2}w\mathbb{I} = \frac{1}{2}\operatorname{grad} q. \tag{60}$$

In particular, applying the tr-operator to both sides of (60), we see that the identity $3w = \operatorname{div} q$ holds on all $\mathbb{R}^3$, so $q \in \mathring{H}(\operatorname{div})$. Furthermore, applying dev-operator to both sides of (60), we conclude that $\tau = \operatorname{dev}\tau = \operatorname{dev}\operatorname{grad} q$. The map $\tau \mapsto q$ is the required operator R_{gc} and its stated norm bound follows from the continuity of T_{g}. □

Lemma 3.18. *There is a linear map* $R_{\mathrm{cc}} : \ker(\operatorname{div}\operatorname{div} : \mathring{\tilde{H}}_{\mathrm{dd}}) \to L_2 \otimes \mathbb{T}$ *such that for all* $\sigma \in \ker(\operatorname{div}\operatorname{div} : \mathring{\tilde{H}}_{\mathrm{dd}})$,

$$\operatorname{sym}\operatorname{curl} R_{\mathrm{cc}}\sigma = \sigma, \qquad \|R_{\mathrm{cc}}\sigma\|_{L_2} \lesssim \|\sigma\|_{\mathring{\tilde{H}}^{-1}}.$$

Proof. By Lemma 3.6, $\sigma - \operatorname{sym}\operatorname{curl}\tilde{R}_{\mathrm{cc}}\sigma$ has vanishing divergence. Applying T_{c} to its rows and using (43c), we obtain a $\rho \in L_2 \otimes \mathbb{M}$ such that $\sigma - \operatorname{sym}\operatorname{curl}\tilde{R}_{\mathrm{cc}}\sigma = \operatorname{curl}\rho$. Hence

$$\begin{aligned} \sigma &= \operatorname{sym}\operatorname{curl}\tilde{R}_{\mathrm{cc}}\sigma + \operatorname{curl}(\operatorname{dev}\rho + \frac{1}{3}\operatorname{tr}(\rho)\mathbb{I}) \\ &= \operatorname{sym}\operatorname{curl}\tilde{R}_{\mathrm{cc}}\sigma + \operatorname{curl}(\operatorname{dev}\rho) - \frac{1}{3}\operatorname{mskw}\operatorname{grad}\operatorname{tr}(\rho) \end{aligned}$$

by (26b). Applying sym-operator to both sides, $\sigma = \operatorname{sym}\operatorname{curl}(\tilde{R}_{\mathrm{cc}}\sigma + \operatorname{dev}\rho)$. The linear map $\sigma \mapsto \tilde{R}_{\mathrm{cc}}\sigma + \operatorname{dev}\rho$ is the required map R_{cc}. □

Lemma 3.19. *There is a linear map* $R_{\mathrm{cd}} : \mathring{\tilde{H}}^{-1}_{\mathcal{ND}}(\operatorname{div}) \to L_2 \otimes \mathbb{S}$ *such that for all* q *in* $\mathring{\tilde{H}}^{-1}_{\mathcal{ND}}(\operatorname{div})$,

$$q = \operatorname{div} R_{\mathrm{cd}} q, \qquad \|R_{\mathrm{cd}} q\|_{L_2} \lesssim \|q\|_{\mathring{\tilde{H}}^{-1}_{\mathcal{ND}}(\operatorname{div})}. \tag{61}$$

Proof. Let $q \in \mathcal{D}_{\mathcal{ND}}$. Since every component of q has zero mean, by (43d), there exists a $\gamma \in L_2(\mathbb{R}^3) \otimes \mathbb{M}$, supported on $\bar{\Omega}$, such that $\operatorname{div}\gamma = q$. In particular, this implies that each row vector of $\gamma|_{\Omega}$ is in $\mathring{H}(\operatorname{div})$. Hence, integration by parts shows that for any $b \in \mathbb{V}$,

$$(q, b \times x) = (\operatorname{div}\gamma, b \times x) = -(\gamma, \operatorname{grad}(b \times x)) = -(\gamma, \operatorname{mskw} b).$$

Since $(q, b \times x) = 0$ for all $q \in \mathcal{D}_{\mathcal{ND}}$, all terms above vanish, so $0 = (\gamma, \operatorname{mskw} b) = (\operatorname{mskw} u, \operatorname{mskw} b)$ with $u = \operatorname{vskw}\gamma$, or equivalently, $(u, b) = 0$. Therefore, applying (43d) to each component of $u \in L_2 \otimes \mathbb{V}$, we obtain a $\tau \in \mathring{H}^1 \otimes \mathbb{M}$ such that $\operatorname{div}\tau = -2u$.

Collecting these observations, and putting $\sigma = \operatorname{sym}\gamma$,

$$\begin{aligned} q = \operatorname{div}\gamma &= \operatorname{div}(\operatorname{sym}\gamma + \operatorname{mskw} u) \\ &= \operatorname{div}\sigma - \operatorname{curl} u && \text{by (26a)}, \\ &= \operatorname{div}\sigma + \frac{1}{2}\operatorname{curl}\operatorname{div}\tau \\ &= \operatorname{div}\sigma + \frac{1}{2}\operatorname{curl}\operatorname{div}\operatorname{dev}\tau \\ &= \operatorname{div}(\sigma + \operatorname{sym}\operatorname{curl}\top\operatorname{dev}\tau) && \text{by (30)}. \end{aligned}$$

Set $R_{\rm cd} q = \sigma + \operatorname{sym}\operatorname{curl}\top\operatorname{dev}\tau$. Then by the continuity of $T_{\rm d}$ in (43d), the norm estimate in (61) follows for any $q \in \mathcal{D}_{\mathcal{ND}}$. The proof is finished using the density result of Lemma 2.2. ☐

Lemma 3.20. *There are continuous linear maps* $R_{\rm g} : \ker(\operatorname{curl} : \mathring{\tilde{H}}^{-1}_{\mathcal{RT}}(\operatorname{curl})) \to L_{2,\mathbb{R}}$, $R_{\rm c} : \ker(\operatorname{div} : \mathring{\tilde{H}}^{-1}_{\mathcal{ND}}(\operatorname{div})) \to (L_2 \otimes \mathbb{V}) \cap \mathring{\tilde{H}}^{-1}_{\mathcal{RT}}(\operatorname{curl})$, *and* $R_{\rm d} : \mathring{\tilde{H}}^{-1}_{\mathcal{P}_1} \to (L_2 \otimes \mathbb{V}) \cap \mathring{\tilde{H}}^{-1}_{\mathcal{ND}}(\operatorname{div})$, *such that for any* $v \in \mathring{\tilde{H}}^{-1}_{\mathcal{RT}}(\operatorname{curl})$ *with* $\operatorname{curl} v = 0$, $q \in \mathring{\tilde{H}}^{-1}_{\mathcal{ND}}(\operatorname{div})$ *with* $\operatorname{div} q = 0$, *and* $w \in \mathring{\tilde{H}}^{-1}_{\mathcal{P}_1}$, *we have*

$$\begin{aligned} &\frac{1}{3}\operatorname{grad} R_{\rm g} v = v, && \frac{1}{2}\operatorname{curl} R_{\rm c} q = q, && \operatorname{div} R_{\rm d} w = w, \\ &\|R_{\rm g} v\|_{L_2} \lesssim \|v\|_{\mathring{\tilde{H}}^{-1}} && \|R_{\rm c} q\|_{L_2} \lesssim \|q\|_{\mathring{\tilde{H}}^{-1}}, && \|R_{\rm d} w\|_{L_2} \lesssim \|w\|_{\mathring{\tilde{H}}^{-1}}. \end{aligned}$$

Proof. Let us construct the last operator first. The functional action of any $w \in \mathring{\tilde{H}}^{-1}_{\mathcal{P}_1}$ on constant functions vanish, so we use (43d) to conclude that $T_{\rm d} w \in L_2 \otimes \mathbb{V}$ satisfies $\operatorname{div} T_{\rm d} w = w$.

We proceed to correct $T_{\rm d}$ to obtain orthogonality to $\mathcal{ND}$. Let $u \in \mathring{H}(\operatorname{curl})$ satisfy

$$(\operatorname{curl} u, \operatorname{curl} v) = (T_{\rm d} w, \operatorname{curl} v), \qquad \operatorname{div} u = 0,$$

for all $v \in \mathring{H}(\operatorname{curl})$, a constrained formulation that is well known to be uniquely solvable [31]. The first equation above implies that

$$\|\operatorname{curl} u\|_{L_2} \le \|T_{\rm d} w\|_{L_2} \lesssim \|w\|_{\mathring{\tilde{H}}^{-1}}, \tag{62}$$

$$\operatorname{curl}\operatorname{curl} u = \operatorname{curl} T_{\rm d} w. \tag{63}$$

Put $R_{\rm d} w := T_{\rm d} w - \operatorname{curl} u \in L_2 \otimes \mathbb{V}$. It satisfies

$$\operatorname{curl} R_{\rm d} w = 0, \qquad \operatorname{div} R_{\rm d} w = w, \qquad (R_{\rm d} w, r) = 0$$

for all $r \in \mathcal{ND}$. Indeed, the first equation follows from (63) and the second from $\operatorname{div} T_{\rm d} w = w$. To see the third, first note that since the functional action of w on any $p_1 \in \mathcal{P}_1$ vanish,

$$0 = w(p_1) = (\operatorname{div} T_{\rm d} w)(p_1) = (T_{\rm d} w, \operatorname{grad} p_1),$$

so $(T_{\rm d} w, a) = 0$ for any $a \in \mathbb{V}$. Combined with $(\operatorname{curl} u, a) = (u, \operatorname{curl} a) = 0$, we have $(R_{\rm d} w, a) = 0$. Moreover, since $\operatorname{curl}(bx \cdot x) = -2b \times x$ for any $b \in \mathbb{V}$, we have

$$0 = (\operatorname{curl} R_{\rm d} w)(bx \cdot x) = (R_{\rm d} w, \operatorname{curl}(bx \cdot x)) = (R_{\rm d} w, -2b \times x),$$

so $R_{\rm d} w$ is L_2-orthogonal to $a + b \times x$ for any $a, b \in \mathbb{V}$. By the continuity of $T_{\rm d}$ and (62), we also obtain the norm bound $\|R_{\rm d} w\|_{L_2} \lesssim \|w\|_{\mathring{\tilde{H}}^{-1}}$.

The construction of $R_{\rm c}$ is similar: set $R_{\rm c} q := 2(T_{\rm c} q - \operatorname{grad} w)$ where $w \in \mathring{H}^1$ solves $(\operatorname{grad} w, \operatorname{grad} z) = (T_{\rm c} q, \operatorname{grad} z)$ for all $z \in \mathring{H}^1$. Clearly, $\frac{1}{2}\operatorname{curl} R_{\rm c} q = q$. Also, for any $a \in \mathbb{V}$, since the action of q on any $\mathcal{ND}$-function vanishes, $0 = q(a\times x) = (\operatorname{curl} T_{\rm c} q)(a\times x) = (T_{\rm c} q, 2a)$, so $(R_{\rm c} q, a) = 0$. Moreover, for any $b \in \mathbb{R}$,

$$(R_{\rm c} q, a + bx) = (R_{\rm c} q, bx) = \frac{1}{2}(R_{\rm c} q, \operatorname{grad}(bx\cdot x)) = \frac{1}{2}(\operatorname{div} R_{\rm c} q)(b|x|^2)$$

which must vanish since $\operatorname{div} R_{\rm c} q = 0$ by construction, so $R_{\rm c} q$ is L_2-orthogonal to $\mathcal{RT}$.

Finally, simply setting $R_{\rm g} = 3T_{\rm g}$, it is easy to verify the stated properties of $R_{\rm g}$. □

Theorem 3.21. *Each R and D operator in* (41) *is a continuous right inverse of the differential operator marked above it.*

Proof. The result is proved by prior lemmas, as seen by the following pointers, which consider the operators, row by row, from left to right, but omitting the obvious ones in the first row and any obvious symmetrically opposite ones.

- $D_{\rm gg} : \ker(\operatorname{curl} : \mathring{\tilde{H}}_{\rm cc}) \to \mathring{H}^1$ is a continuous right inverse of $\operatorname{hess} : \mathring{H}(\operatorname{grad}) \to \mathring{\tilde{H}}_{\rm cc}$ by Lemma 3.3 applied with $s = -1$.
- $D_{\rm gc} : \ker(\operatorname{div} : \mathring{\tilde{H}}_{\rm cd}) \cap \ker(\operatorname{sym}\operatorname{curl}\top : \mathring{\tilde{H}}_{\rm cd}) \to \mathring{H}^1 \otimes \mathbb{V} \subset \mathring{H}(\operatorname{curl})$ is a continuous right inverse of $\operatorname{curl}\operatorname{def} : \mathring{H}(\operatorname{curl}) \to \mathring{\tilde{H}}_{\rm cd}$ by Lemma 3.13.
- $D_{\rm gd} : \ker(\operatorname{curl} : \mathring{\tilde{H}}^{-1}_{\mathcal{RT}}(\operatorname{curl})) \to \mathring{H}^1 \otimes \mathbb{V} \subset \mathring{H}(\operatorname{div})$ is a continuous right inverse of $\frac{1}{3}\operatorname{grad}\operatorname{div} : \mathring{H}(\operatorname{div}) \to \mathring{\tilde{H}}^{-1}_{\mathcal{RT}}(\operatorname{curl})$ by Lemma 3.14.
- $R_{\rm gg} : \ker(\operatorname{inc} : \mathring{\tilde{H}}_{\rm cc}) \to \mathring{H}(\operatorname{curl})$ is a continuous right inverse of $\operatorname{def} : \mathring{H}(\operatorname{curl}) \to \mathring{\tilde{H}}_{\rm cc}$ by Lemma 3.15.
- $R_{\rm gc} : \ker(\operatorname{div} : \mathring{\tilde{H}}_{\rm cd}) \to L_2 \otimes \mathbb{S} \subset \mathring{\tilde{H}}_{\rm cc}$ is a continuous right inverse of $\operatorname{curl} : \mathring{\tilde{H}}_{\rm cc} \to \mathring{\tilde{H}}_{\rm cd}$ by Lemma 3.12.
- $R_{\rm gd} : \mathring{\tilde{H}}^{-1}_{\mathcal{RT}}(\operatorname{curl}) \to L_2 \otimes \mathbb{T} \subset \mathring{\tilde{H}}_{\rm cd}$ is a continuous right inverse of $\operatorname{div} : \mathring{\tilde{H}}_{\rm cd} \to \mathring{\tilde{H}}^{-1}_{\mathcal{RT}}(\operatorname{curl})$ by Lemma 3.16.
- $R_{\rm gc\top} : \ker(\operatorname{sym}\operatorname{curl} : \mathring{\tilde{H}}_{\rm cd\top}) \to \mathring{H}(\operatorname{div})$ is a continuous right inverse of $\frac{1}{2}\operatorname{dev}\operatorname{grad} : \mathring{H}(\operatorname{div}) \to \mathring{\tilde{H}}_{\rm cd\top}$ by Lemma 3.17.
- $D_{\rm cc} : \ker(\operatorname{div} : \mathring{\tilde{H}}_{\rm dd}) \to \mathring{H}^1 \otimes \mathbb{S}$ is a continuous right inverse of $\operatorname{inc} : \mathring{\tilde{H}}_{\rm cc} \to \mathring{\tilde{H}}_{\rm dd}$ by Lemma 3.1 applied with $s = -1$.
- $D_{\rm cd} : \ker(\operatorname{div} : \mathring{\tilde{H}}^{-1}_{\mathcal{ND}}(\operatorname{div})) \to \mathring{H}^1 \otimes \mathbb{T} \subset \mathring{\tilde{H}}_{\rm dd}$ is a continuous right inverse of $\frac{1}{2}\operatorname{curl}\operatorname{div} : \mathring{\tilde{H}}_{\rm cd} \to \mathring{\tilde{H}}^{-1}_{\mathcal{ND}}(\operatorname{div})$ by Lemma 3.9 applied with $s = -1$.
- $R_{\rm cc} : \ker(\operatorname{div}\operatorname{div} : \mathring{\tilde{H}}_{\rm dd}) \to L_2 \otimes \mathbb{T} \subset \mathring{\tilde{H}}_{\rm cd\top}$ is a continuous right inverse of $\operatorname{sym}\operatorname{curl} : \mathring{\tilde{H}}_{\rm cd\top} \to \mathring{\tilde{H}}_{\rm dd}$ by Lemma 3.18.
- $R_{\rm cd} : \mathring{\tilde{H}}^{-1}_{\mathcal{ND}}(\operatorname{div}) \to L_2 \otimes \mathbb{S} \subset \mathring{\tilde{H}}_{\rm dd}$ is a continuous right inverse of $\operatorname{div} : \mathring{\tilde{H}}_{\rm cd} \to \mathring{\tilde{H}}^{-1}_{\mathcal{ND}}(\operatorname{div})$ by Lemma 3.19.
- $D_{\rm dd} : \mathring{\tilde{H}}^{-1}_{\mathcal{P}_1} \to \mathring{H}^1 \otimes \mathbb{S} \subset \mathring{\tilde{H}}_{\rm dd}$ is a continuous right inverse of $\operatorname{div}\operatorname{div} : \mathring{\tilde{H}}_{\rm dd} \to \mathring{\tilde{H}}^{-1}_{\mathcal{P}_1}$ by Lemma 3.5 applied with $s = -1$.
- $R_{\rm g} : \ker(\operatorname{curl} : \mathring{\tilde{H}}^{-1}_{\mathcal{RT}}(\operatorname{curl})) \to L_{2,\mathbb{R}}$, $R_{\rm c} : \ker(\operatorname{div} : \mathring{\tilde{H}}^{-1}_{\mathcal{ND}}(\operatorname{div})) \to (L_2 \otimes \mathbb{V}) \cap \mathring{\tilde{H}}^{-1}_{\mathcal{RT}}(\operatorname{curl})$, and $R_{\rm d} : \mathring{\tilde{H}}^{-1}_{\mathcal{P}_1} \to (L_2 \otimes \mathbb{V}) \cap \mathring{\tilde{H}}^{-1}_{\mathcal{ND}}(\operatorname{div})$, are continuous right inverses of $\frac{1}{3}\operatorname{grad} : L_{2,\mathbb{R}} \to \mathring{\tilde{H}}^{-1}_{\mathcal{RT}}(\operatorname{curl})$, $\frac{1}{2}\operatorname{curl} : \mathring{\tilde{H}}^{-1}_{\mathcal{RT}}(\operatorname{curl}) \to \mathring{\tilde{H}}^{-1}_{\mathcal{ND}}(\operatorname{div})$, and $\operatorname{div} : \mathring{\tilde{H}}^{-1}_{\mathcal{ND}}(\operatorname{div}) \to \mathring{\tilde{H}}^{-1}_{\mathcal{P}_1}$, respectively, by Lemma 3.20.

□

Corollary 3.22. *The range of every differential operator in* (41) *is closed.*

Proof. This is an immediate consequence of the existence of continuous right inverses for each differential operator in (41), as we have shown.

For example, consider the operator curl def . Clearly its range is contained in both $\ker(\operatorname{div} : \mathring{\tilde{H}}_{\mathrm{cd}})$ and $\ker(\operatorname{sym}\operatorname{curl}\top : \mathring{\tilde{H}}_{\mathrm{cd}})$. But Lemma 3.13 shows that the intersection of these kernels is also contained in the range of curl def, so

$$\operatorname{range}(\operatorname{curl}\operatorname{def}) = \ker(\operatorname{div} : \mathring{\tilde{H}}_{\mathrm{cd}}) \cap \ker(\operatorname{sym}\operatorname{curl}\top : \mathring{\tilde{H}}_{\mathrm{cd}}).$$

By the continuity of the differential operators proved in Theorem 2.3, both the above kernels are closed, and so is their intersection. Hence the range of curl def is closed.

The proofs for the remaining operators are similar and simpler. □

Corollary 3.23. *The hessian complex* (38)*, the elasticity complex* (39)*, and the div-div complex* (40) *are exact complexes.*

Proof. To prove that the hessian complex (38) is exact it suffices to prove that $\operatorname{range}(\operatorname{hess}) \supseteq \ker(\operatorname{curl} : \mathring{\tilde{H}}_{\mathrm{cc}})$, $\operatorname{range}(\operatorname{curl}) \supseteq \ker(\operatorname{div} : \mathring{\tilde{H}}_{\mathrm{cd}})$, and $\operatorname{range}(\operatorname{div}) \supseteq \mathring{\tilde{H}}^{-1}_{\mathcal{RT}}(\operatorname{curl})$ (since the reverse inclusions are clear from Corollary 2.5). But these are now obvious by the existence of continuous right inverses $D_{\mathrm{gg}}, R_{\mathrm{gc}}$, and R_{gd} given by Lemmas 3.3, 3.12, and 3.14, respectively.

Similarly, the continuous right inverse operators R_{gg}, D_{cc}, and R_{cd}, given by Lemmas 3.15, 3.1, and 3.19, prove the exactness of the elasticity complex.

The exactness of the div div complex similarly follows from the continuous right inverse operators of $R_{\mathrm{gc}\top}, R_{\mathrm{cc}}$ and D_{dd} of Lemmas 3.17, 3.18, and 3.5. □

4. Slightly more regular spaces of matrix fields

Consider the following slightly more regular spaces of matrix fields, contained in the previously introduced spaces $\mathring{\tilde{H}}_{\mathrm{cc}}$, $\mathring{\tilde{H}}_{\mathrm{cd}}$ and $\mathring{\tilde{H}}_{\mathrm{dd}}$:

$$\mathring{\tilde{\mathcal{H}}}_{\mathrm{cc}} = \{g \in L_2 \otimes \mathbb{S} : \operatorname{inc} g \in \mathring{\tilde{H}}^{-1} \otimes \mathbb{S}\}, \tag{64}$$

$$\mathring{\tilde{\mathcal{H}}}_{\mathrm{cd}} = \{\tau \in L_2 \otimes \mathbb{T} : \operatorname{curl}\operatorname{div}\tau \in \mathring{\tilde{H}}^{-1} \otimes \mathbb{V}\}, \tag{65}$$

$$\mathring{\tilde{\mathcal{H}}}_{\mathrm{dd}} = \{\sigma \in L_2 \otimes \mathbb{S} : \operatorname{div}\operatorname{div}\sigma \in \mathring{\tilde{H}}^{-1}\}, \tag{66}$$

whose natural norms are defined respectively by

$$\|g\|^2_{\mathring{\tilde{\mathcal{H}}}_{\mathrm{cc}}} = \|g\|^2_{L_2} + \|\operatorname{inc} g\|^2_{\mathring{\tilde{H}}^{-1}},$$
$$\|\tau\|^2_{\mathring{\tilde{\mathcal{H}}}_{\mathrm{cd}}} = \|\tau\|^2_{L_2} + \|\operatorname{curl}\operatorname{div}\tau\|^2_{\mathring{\tilde{H}}^{-1}},$$
$$\|\sigma\|^2_{\mathring{\tilde{\mathcal{H}}}_{\mathrm{dd}}} = \|\sigma\|^2_{L_2} + \|\operatorname{div}\operatorname{div}\sigma\|^2_{\mathring{\tilde{H}}^{-1}}.$$

Such spaces of matrix fields and their even smoother versions, have emerged in recent works [2, 20, 38].

Note that one possible way to increase the regularity of the prior spaces $\mathring{\tilde{H}}_{\mathrm{cc}}$, $\mathring{\tilde{H}}_{\mathrm{cd}}$ and $\mathring{\tilde{H}}_{\mathrm{dd}}$ is to uniformly replace $\mathring{\tilde{H}}^{-1}$ by $\mathring{\tilde{H}}^{s}$ with some $s > -1$ in (23) (and we expect the prior analysis to go through with minimal changes for such modification). The new spaces of this section, $\mathring{\tilde{\mathcal{H}}}_{\mathrm{cc}}, \mathring{\tilde{\mathcal{H}}}_{\mathrm{cd}}$, and $\mathring{\tilde{\mathcal{H}}}_{\mathrm{dd}}$, are not obtained this way. Instead, they are obtained by increasing the regularity of the matrix-valued function to L_2 while maintaining the same $\mathring{\tilde{H}}^{-1}$-regularity for its second-order derivative. We proceed to prove regular decompositions and density of smooth functions for such spaces.

Theorem 4.1 (Regular decomposition of $\mathring{\widetilde{\mathcal{H}}}_{\mathrm{cc}}$)**.** *There exist continuous linear operators $\mathring{S}^{(0)}_{\mathrm{cc}} : \mathring{\widetilde{\mathcal{H}}}_{\mathrm{cc}} \to \mathring{H}^1 \otimes \mathbb{S}$, $\mathring{\mathcal{S}}^{(1)}_{\mathrm{cc}} : \mathring{\widetilde{\mathcal{H}}}_{\mathrm{cc}} \to \mathring{H}^1 \otimes \mathbb{V}$ such that any $g \in \mathring{\widetilde{\mathcal{H}}}_{\mathrm{cc}}$ can be decomposed into*

$$g = \mathring{S}^{(0)}_{\mathrm{cc}}\, g + \operatorname{def} \mathring{\mathcal{S}}^{(1)}_{\mathrm{cc}}\, g. \tag{67}$$

Consequently,

$$\mathring{\widetilde{\mathcal{H}}}_{\mathrm{cc}} = \overline{\mathcal{D}(\Omega) \otimes \mathbb{S}}^{\|\cdot\|_{\widetilde{\mathcal{H}}_{\mathrm{cc}}}}. \tag{68}$$

Proof. Since any $g \in \mathring{\widetilde{\mathcal{H}}}_{\mathrm{cc}}$ in also in $\mathring{\widetilde{H}}_{\mathrm{cc}}$, we apply the operators $\mathring{S}^{(0)}_{\mathrm{cc}}$ and $\mathring{S}^{(1)}_{\mathrm{cc}}$ of Theorem 3.4 and follow along the lines of its proof to obtain (46), $\operatorname{curl}\big(g - \mathring{S}^{(0)}_{\mathrm{cc}}\, g - \operatorname{def} \mathring{S}^{(1)}_{\mathrm{cc}}\, g\big) = 0$. Setting $\mathring{S}^{(2)}_{\mathrm{cc}}\, g = D_{\mathrm{gg}}(g - \mathring{S}^{(0)}_{\mathrm{cc}}\, g - \operatorname{def} \mathring{S}^{(1)}_{\mathrm{cc}}\, g)$ we apply Lemma 3.3, but this time with $s = 0$ since g is now in $L_2 \otimes \mathbb{S}$, to get that

$$\mathring{S}^{(2)}_{\mathrm{cc}}\, g \in \mathring{\widetilde{H}}^2 \otimes \mathbb{S}, \qquad g - \mathring{S}^{(0)}_{\mathrm{cc}}\, g - \operatorname{def} \mathring{S}^{(1)}_{\mathrm{cc}}\, g = \operatorname{hess} \mathring{S}^{(2)}_{\mathrm{cc}}\, g.$$

Now (67) follows setting $\mathring{\mathcal{S}}^{(1)}_{\mathrm{cc}}\, g = \mathring{S}^{(1)}_{\mathrm{cc}}\, g + \operatorname{grad} \mathring{S}^{(2)}_{\mathrm{cc}}\, g$.

To prove (68), decompose $g \in \mathring{\widetilde{\mathcal{H}}}_{\mathrm{cc}}$ as above. By the density of $\mathcal{D}(\Omega)$ in $\mathring{H}^1$, there are $g_m \in \mathcal{D}(\Omega) \otimes \mathbb{S}$ and $u_m \in \mathcal{D}(\Omega) \otimes \mathbb{V}$ such that

$$\|g_m - \mathring{S}^{(0)}_{\mathrm{cc}}\, g\|_{H^1} \to 0, \qquad \|u_m - \mathring{S}^{(1)}_{\mathrm{cc}}\, g\|_{H^1} \to 0,$$

as $m \to \infty$. Hence, $g_m + \operatorname{def} u_m \in \mathcal{D}(\Omega) \otimes \mathbb{S}$ and

$$\|g - (g_m + \operatorname{def} u_m)\|^2_{\widetilde{\mathcal{H}}_{\mathrm{cc}}} \le \|g - g_m\|^2_{L_2} + \|\operatorname{inc}(g - g_m)\|^2_{\mathring{\widetilde{H}}^{-1}} \lesssim \|g - g_m\|^2_{H^1}$$

by (18). Since the last term converges to zero, we have proved that $\mathring{\widetilde{\mathcal{H}}}_{\mathrm{cc}}$ is contained in the closure of $\mathcal{D}(\Omega) \otimes \mathbb{S}$. The reverse inclusion is obvious. □

Theorem 4.2 (Regular decomposition of $\mathring{\widetilde{\mathcal{H}}}_{\mathrm{dd}}$)**.** *There exist continuous linear operators $\mathring{S}^{(0)}_{\mathrm{dd}} : \mathring{\widetilde{H}}_{\mathrm{dd}} \to \mathring{H}^1 \otimes \mathbb{S}$ and $\mathring{\mathcal{S}}^{(1)}_{\mathrm{dd}} : \mathring{\widetilde{H}}_{\mathrm{dd}} \to \mathring{H}^1 \otimes \mathbb{T}$ such that any $\sigma \in \mathring{\widetilde{\mathcal{H}}}_{\mathrm{dd}}$ can be decomposed into*

$$\sigma = \mathring{S}^{(0)}_{\mathrm{dd}}\, \sigma + \operatorname{sym}\operatorname{curl} \mathring{\mathcal{S}}^{(1)}_{\mathrm{dd}}\, \sigma. \tag{69}$$

Consequently,

$$\mathring{\widetilde{\mathcal{H}}}_{\mathrm{dd}} = \overline{\mathcal{D}(\Omega) \otimes \mathbb{S}}^{\|\cdot\|_{\widetilde{\mathcal{H}}_{\mathrm{dd}}}}. \tag{70}$$

Proof. We proceed along the lines of the proof of Theorem 3.7, but now with the more regular σ in $\mathring{\widetilde{\mathcal{H}}}_{\mathrm{dd}}$, to obtain that

$$\operatorname{div}\big(\sigma - \mathring{S}^{(0)}_{\mathrm{dd}}\, \sigma - \operatorname{sym}\operatorname{curl} \mathring{S}^{(1)}_{\mathrm{dd}}\, \sigma\big) = 0,$$

with $\mathring{S}^{(0)}_{\mathrm{dd}}$ and $\mathring{S}^{(1)}_{\mathrm{dd}}$ as defined there. At this point, we apply Lemma 3.1, now with $s = 0$, setting $\mathring{S}^{(2)}_{\mathrm{dd}}\, \sigma := D_{\mathrm{cc}}(\sigma - \mathring{S}^{(0)}_{\mathrm{dd}}\, \sigma - \operatorname{sym}\operatorname{curl} \mathring{S}^{(1)}_{\mathrm{dd}}\, \sigma)$, to find that

$$\mathring{S}^{(2)}_{\mathrm{cc}}\, \sigma \in \mathring{\widetilde{H}}^2 \otimes \mathbb{S}, \qquad \sigma - \mathring{S}^{(0)}_{\mathrm{dd}}\, \sigma - \operatorname{sym}\operatorname{curl} \mathring{S}^{(1)}_{\mathrm{dd}}\, \sigma = \operatorname{inc} \mathring{S}^{(2)}_{\mathrm{dd}}\, \sigma.$$

The result now follows setting $\mathring{\mathcal{S}}^{(1)}_{\mathrm{dd}}\, \sigma = \mathring{S}^{(1)}_{\mathrm{dd}}\, \sigma + {}^\top\!\operatorname{curl} \mathring{S}^{(2)}_{\mathrm{dd}}\, \sigma$.

To conclude, let $\sigma \in \mathring{\widetilde{\mathcal{H}}}_{\mathrm{dd}}$ and use the just proved decomposition (69) to split it into $\sigma = \mathring{S}^{(0)}_{\mathrm{dd}}\, \sigma + \operatorname{sym}\operatorname{curl} \mathring{\mathcal{S}}^{(1)}_{\mathrm{dd}}\, \sigma$. By the density of $\mathcal{D}(\Omega)$ in $\mathring{H}^1$, there are $\sigma_m \in \mathcal{D}(\Omega) \otimes \mathbb{S}$ and $\tau_m \in \mathcal{D}(\Omega) \otimes \mathbb{T}$ such that

$$\|\sigma_m - \mathring{S}^{(0)}_{\mathrm{dd}}\, \sigma\|_{H^1} \to 0, \qquad \|\tau_m - \mathring{\mathcal{S}}^{(1)}_{\mathrm{dd}}\, \sigma\|_{H^1} \to 0,$$

as $m \to \infty$. Hence, by (18) and (30) $\sigma_m + \operatorname{sym}\operatorname{curl}\tau_m \in \mathcal{D}(\Omega)\otimes\mathbb{S}$ converges to σ in $\|\cdot\|_{\mathring{\tilde{\mathcal{H}}}_{\rm dd}}$ norm, thus proving that σ is contained in the closure of $\mathcal{D}(\Omega)\otimes\mathbb{S}$. Since the reverse inclusion is obvious, (70) is proved. □

Theorem 4.3 (Regular decomposition of $\mathring{\tilde{\mathcal{H}}}_{\rm cd}$). *There exist three continuous linear operators*

$$\mathring{S}^{(0)}_{\rm cd} : \mathring{\tilde{\mathcal{H}}}_{\rm cd} \to \mathring{H}^1\otimes\mathbb{T}, \quad \mathring{S}^{(1)}_{\rm cd} : \mathring{\tilde{\mathcal{H}}}_{\rm cd} \to \mathring{H}^1\otimes\mathbb{S}, \quad \mathring{\mathcal{S}}^{(2)}_{\rm cd} : \mathring{\tilde{\mathcal{H}}}_{\rm cd} \to \mathring{H}^1\otimes\mathbb{V},$$

such that any $\tau\in\mathring{\tilde{\mathcal{H}}}_{\rm cd}$ can be decomposed into

$$\tau = \mathring{S}^{(0)}_{\rm cd}\,\tau + \operatorname{curl}\mathring{S}^{(1)}_{\rm cd}\,\tau + \top\operatorname{dev}\operatorname{grad}\mathring{\mathcal{S}}^{(2)}_{\rm cd}\,\tau. \tag{71}$$

Consequently,

$$\mathring{\tilde{\mathcal{H}}}_{\rm cd} = \overline{\mathcal{D}(\Omega)\otimes\mathbb{T}}^{\|\cdot\|_{\mathring{\tilde{\mathcal{H}}}_{\rm cd}}}. \tag{72}$$

Proof. We follow along the lines of the proof of Theorem 3.10 to find that given any $\tau\in\mathring{\tilde{\mathcal{H}}}_{\rm cd}$, using the same $\mathring{S}^{(0)}_{\rm cd}$ operator defined there, we have $\operatorname{curl}\operatorname{div}(\tau - \mathring{S}^{(0)}_{\rm cd}\,\tau) = 0$. But now, since the higher regularity of τ implies that $\tau - \mathring{S}^{(0)}_{\rm cd}\,\tau$ is in $L_2\otimes\mathbb{T}$, we may apply the two-term decomposition of Lemma 3.8 (taking $\hat{U}_{\rm gc}$ there to be zero) instead of the three-term decomposition to obtain

$$\tau - \mathring{S}^{(0)}_{\rm cd}\,\tau = (\operatorname{curl}\hat{R}_{\rm gc} + \top\operatorname{dev}\operatorname{grad}\hat{D}_{\rm gc})(\tau - \mathring{S}^{(0)}_{\rm cd}\,\tau).$$

The decomposition (71) now follows after setting $\mathring{S}^{(1)}_{\rm cd}\,\tau = \hat{R}_{\rm gc}(\tau - \mathring{S}^{(0)}_{\rm cd}\,\tau)$, and $\mathring{\mathcal{S}}^{(2)}_{\rm cd}\,\tau = \hat{D}_{\rm gc}(\tau - \mathring{S}^{(0)}_{\rm cd}\,\tau)$.

To prove (72), let $\tau\in\mathring{\tilde{\mathcal{H}}}_{\rm cd}$ be decomposed as in (71). By the density of $\mathcal{D}(\Omega)$ in $\mathring{H}^1$, there are $\tau_m\in\mathcal{D}(\Omega)\otimes\mathbb{T}$, $g_m\in\mathcal{D}(\Omega)\otimes\mathbb{S}$ and $q_m\in\mathcal{D}(\Omega)\otimes\mathbb{V}$ which converge to the decomposition components $\mathring{S}^{(0)}_{\rm cd}\,\tau$, $\mathring{S}^{(1)}_{\rm cd}\,\tau$ and $\top\operatorname{dev}\operatorname{grad}\mathring{\mathcal{S}}^{(2)}_{\rm cd}\,\tau$, respectively, in the H^1-norm, as $m\to\infty$. Hence

$$\begin{aligned}
&\|\tau_m + \operatorname{curl} g_m + \top\operatorname{dev}\operatorname{grad} q_m - \tau\|_{\mathring{\tilde{\mathcal{H}}}_{\rm cd}} \\
&\quad\le \|\tau_m - \mathring{S}^{(0)}_{\rm cd}\,\tau\|_{\mathring{\tilde{\mathcal{H}}}_{\rm cd}} + \|\operatorname{curl}(g_m - \mathring{S}^{(1)}_{\rm cd}\,\tau)\|_{\mathring{\tilde{\mathcal{H}}}_{\rm cd}} + \|\top\operatorname{dev}\operatorname{grad}(q_m - \mathring{\mathcal{S}}^{(2)}_{\rm cd}\,\tau)\|_{\mathring{\tilde{\mathcal{H}}}_{\rm cd}} \\
&\quad\lesssim \|\tau_m - \mathring{S}^{(0)}_{\rm cd}\,\tau\|_{H^1} + \|\operatorname{curl}(g_m - \mathring{S}^{(1)}_{\rm cd}\,\tau)\|_{L_2} + \|\top\operatorname{dev}\operatorname{grad}(q_m - \mathring{\mathcal{S}}^{(2)}_{\rm cd}\,\tau)\|_{L_2} \\
&\quad\lesssim \|\tau_m - \mathring{S}^{(0)}_{\rm cd}\,\tau\|_{H^1} + \|g_m - \mathring{S}^{(1)}_{\rm cd}\,\tau\|_{H^1} + \|q_m - \mathring{\mathcal{S}}^{(2)}_{\rm cd}\,\tau\|_{H^1}
\end{aligned}$$

where we have used (31) (which implies that $\operatorname{curl}\operatorname{div}\circ\top\operatorname{dev}\operatorname{grad} = 0$) and (18). Since the last bound converges to zero as $m\to\infty$, we have just exhibited a sequence in $\mathcal{D}(\Omega)\otimes\mathbb{T}$ that approximates any given $\tau\in\mathring{\tilde{\mathcal{H}}}_{\rm cd}$ arbitrarily closely in $\mathring{\tilde{\mathcal{H}}}_{\rm cd}$ norm. □

5. Duality

In this section, we state extensions to 2-complexes built using Sobolev spaces that are generally not closures of compactly supported smooth functions, such as those in (14), as well as spaces built using H^{-1} instead of $\mathring{\tilde{H}}^{-1}$, such as

$$H^{-1}(\operatorname{curl}) = \{u\in H^{-1}\otimes\mathbb{V} : \operatorname{curl} u\in H^{-1}\otimes\mathbb{V}\},$$
$$H^{-1}(\operatorname{div}) = \{q\in H^{-1}\otimes\mathbb{V} : \operatorname{div} q\in H^{-1}\},$$

spaces of matrix-valued functions,

$$H_{\mathrm{cc}} = \{g \in H^{-1} \otimes \mathbb{S} : \operatorname{curl} g \in H^{-1} \otimes \mathbb{V}, \operatorname{inc} g \in H^{-1} \otimes \mathbb{S}\}, \tag{73a}$$

$$H_{\mathrm{cd}} = \{\tau \in H^{-1} \otimes \mathbb{T} : \operatorname{div} \tau \in H^{-1} \otimes \mathbb{V}, \operatorname{sym}\operatorname{curl} \tau \in H^{-1} \otimes \mathbb{S}, \operatorname{curl}\operatorname{div} \tau \in H^{-1} \otimes \mathbb{V}\}, \tag{73b}$$

$$H_{\mathrm{dd}} = \{\sigma \in H^{-1} \otimes \mathbb{S} : \operatorname{div} \sigma \in H^{-1} \otimes \mathbb{V}, \operatorname{div}\operatorname{div} \sigma \in H^{-1}\}, \tag{73c}$$

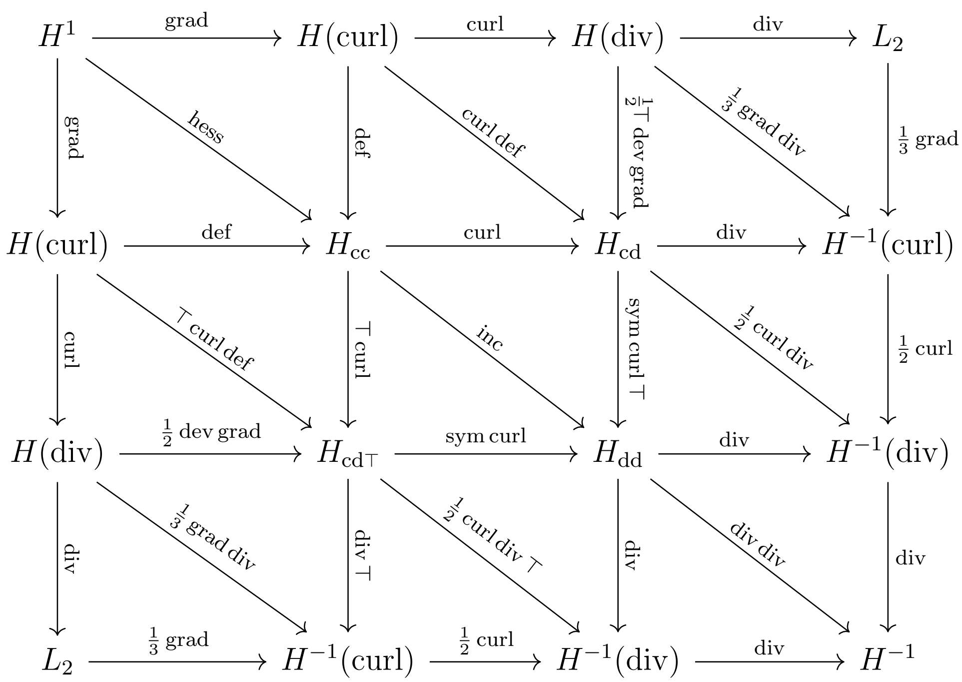
and their slightly more regular versions,

$$\mathcal{H}_{\mathrm{cc}} = \{g \in L_2 \otimes \mathbb{S} : \operatorname{inc} g \in H^{-1} \otimes \mathbb{S}\}, \tag{74a}$$

$$\mathcal{H}_{\mathrm{dd}} = \{\sigma \in L_2 \otimes \mathbb{S} : \operatorname{div}\operatorname{div} \sigma \in H^{-1}\}, \tag{74b}$$

$$\mathcal{H}_{\mathrm{cd}} = \{\tau \in L_2 \otimes \mathbb{T} : \operatorname{curl}\operatorname{div} \tau \in H^{-1} \otimes \mathbb{V}\}. \tag{74c}$$

Analogues of previous results can be proved for



(75)

using exactly the same techniques. They are summarized in the next theorem. We use $\mathcal{D}(\bar{\Omega})$ to denote the space of restrictions of functions in $\mathcal{D}(\mathbb{R}^3)$ to the closure $\bar{\Omega}$.

Theorem 5.1.

(1) *The diagram* (75) *commutes.*
(2) *Every path in it is a 2-complex.*
(3) *Every differential operator in it is continuous, has closed range, and has a continuous right inverse.*
(4) *There are regular decompositions for* $H_{\mathrm{cc}}, H_{\mathrm{dd}}$, *and* H_{cd}. *Namely, there exist continuous linear operators* $S^{(0)}_{\mathrm{cc}} : H_{\mathrm{cc}} \to H^1 \otimes \mathbb{S}$, $S^{(1)}_{\mathrm{cc}} : H_{\mathrm{cc}} \to H^1 \otimes \mathbb{V}$, $S^{(2)}_{\mathrm{cc}} : H_{\mathrm{cc}} \to H^1, S^{(0)}_{\mathrm{dd}} : H_{\mathrm{dd}} \to H^1 \otimes \mathbb{S}$, $S^{(1)}_{\mathrm{dd}} : H_{\mathrm{dd}} \to H^1 \otimes \mathbb{T}$, $S^{(2)}_{\mathrm{dd}} : H_{\mathrm{dd}} \to H^1 \otimes \mathbb{S}, S^{(0)}_{\mathrm{cd}} : H_{\mathrm{cd}} \to H^1 \otimes \mathbb{T}$, $S^{(1)}_{\mathrm{cd}}$:

$H_{\mathrm{cd}} \to H^1 \otimes \mathbb{S},\ S^{(2)}_{\mathrm{cd}} : H_{\mathrm{cd}} \to H^1 \otimes \mathbb{V},\ S^{(3)}_{\mathrm{cd}} : H_{\mathrm{cd}} \to H^1 \otimes \mathbb{V},$ *such that*

$$g = S^{(0)}_{\mathrm{cc}}\, g + \operatorname{def} S^{(1)}_{\mathrm{cc}}\, g + \operatorname{hess} S^{(2)}_{\mathrm{cc}}\, g, \qquad g \in H_{\mathrm{cc}}, \tag{76}$$

$$\sigma = S^{(0)}_{\mathrm{dd}}\, \sigma + \operatorname{sym}\operatorname{curl} S^{(1)}_{\mathrm{dd}}\, \sigma + \operatorname{inc} S^{(2)}_{\mathrm{dd}}\, \sigma, \qquad \sigma \in H_{\mathrm{dd}}, \tag{77}$$

$$\tau = S^{(0)}_{\mathrm{cd}}\, \tau + \operatorname{curl} S^{(1)}_{\mathrm{cd}}\, \tau + \top \operatorname{dev}\operatorname{grad} S^{(2)}_{\mathrm{cd}}\, \tau + \operatorname{curl}\operatorname{def} S^{(3)}_{\mathrm{cd}}\, \tau, \qquad \tau \in H_{\mathrm{cd}}. \tag{78}$$

(5) *The spaces* $\mathcal{D}(\bar\Omega) \otimes \mathbb{S}, \mathcal{D}(\bar\Omega) \otimes \mathbb{T},$ *and* $\mathcal{D}(\bar\Omega) \otimes \mathbb{S}$ *are dense in* $H_{\mathrm{cc}}, H_{\mathrm{cd}},$ *and* H_{dd}*, respectively.*

(6) *There are regular decompositions for* $\mathcal{H}_{\mathrm{cc}}, \mathcal{H}_{\mathrm{dd}},$ *and* $\mathcal{H}_{\mathrm{cd}}$*. Namely, there are continuous linear operators* $\mathcal{S}^{(1)}_{\mathrm{cc}} : \mathcal{H}_{\mathrm{cc}} \to \mathring{H}^1 \otimes \mathbb{V},\ \mathcal{S}^{(2)}_{\mathrm{cd}} : \mathcal{H}_{\mathrm{cd}} \to \mathring{H}^1 \otimes \mathbb{S},\ \mathcal{S}^{(1)}_{\mathrm{dd}} : \mathcal{H}_{\mathrm{dd}} \to \mathring{H}^1 \otimes \mathbb{T},$ *such that*

$$g = S^{(0)}_{\mathrm{cc}}\, g + \operatorname{def} \mathcal{S}^{(1)}_{\mathrm{cc}}\, g, \qquad g \in \mathcal{H}_{\mathrm{cc}}, \tag{79}$$

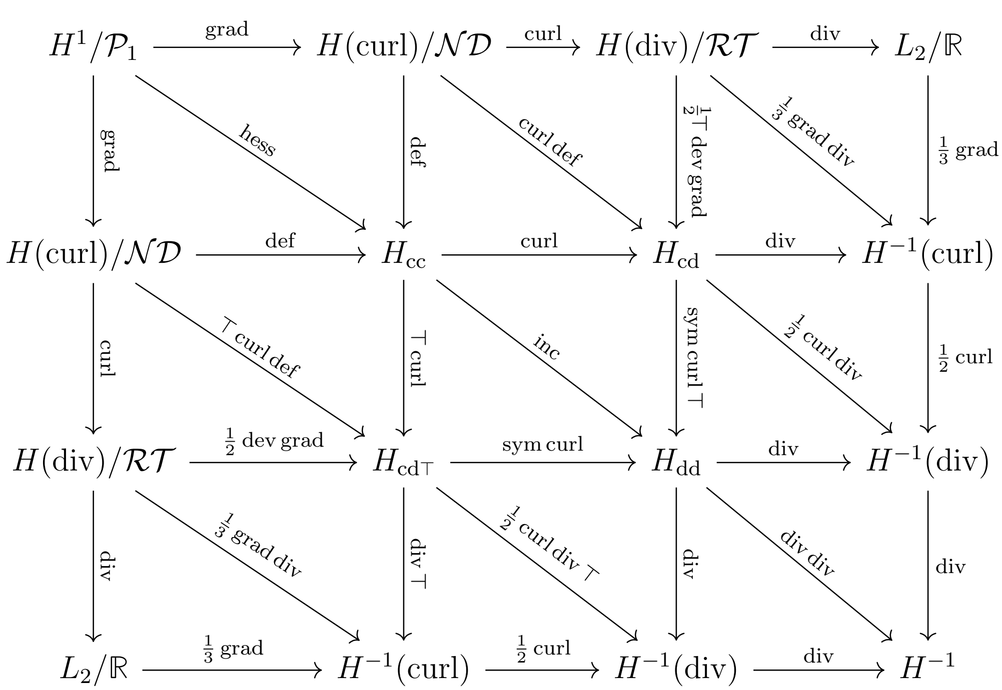
$$\tau = S^{(0)}_{\mathrm{cd}}\, \tau + \operatorname{curl} S^{(1)}_{\mathrm{cd}}\, \tau + \top \operatorname{dev}\operatorname{grad} \mathcal{S}^{(2)}_{\mathrm{cd}}\, \tau, \qquad \tau \in \mathcal{H}_{\mathrm{cd}}, \tag{80}$$

$$\sigma = S^{(0)}_{\mathrm{dd}}\, \sigma + \operatorname{sym}\operatorname{curl} \mathcal{S}^{(1)}_{\mathrm{dd}}\, \sigma, \qquad \sigma \in \mathcal{H}_{\mathrm{dd}}. \tag{81}$$

(7) *The spaces* $\mathcal{D}(\bar\Omega) \otimes \mathbb{S}, \mathcal{D}(\bar\Omega) \otimes \mathbb{T},$ *and* $\mathcal{D}(\bar\Omega) \otimes \mathbb{S}$ *are dense in* $\mathcal{H}_{\mathrm{cc}}, \mathcal{H}_{\mathrm{cd}},$ *and* $\mathcal{H}_{\mathrm{dd}}$*, respectively.*

Proof. Proceed as in the proofs of Theorems 2.3, 2.4, and 3.21. □

Some spaces at the edges of the diagram (75) can be restricted to certain subspaces of interest without affecting the matrix-valued function spaces to get the following commuting diagram:



(82)

This is because of the following facts: (a) $\operatorname{grad} \mathcal{P}_1 = \mathbb{V}$ is in the zero coset of $H(\mathrm{curl})/\mathcal{ND}$, (b) $\operatorname{curl} \mathcal{ND} = \mathbb{V}$ is in the zero coset of $H(\mathrm{div})/\mathcal{RT}$, (c) $\operatorname{div} \mathcal{RT} = \mathbb{R}$ is in the zero coset of $L_2/\mathbb{R}$, (d) $\operatorname{def} \mathcal{ND} = 0$, and (e) $\operatorname{dev}\operatorname{grad} \mathcal{RT} = 0$.

Next we turn to establishing certain duality relationships between the spaces in the diagram (25) and those in the just introduced diagram (82). First we need a lemma that enlarges the domain of certain functionals. Consider a $q \in H^{-1}(\mathrm{div})$. Then, since q is in $H^{-1} \otimes \mathbb{V}$, its action on a function in $\mathring{H}^1 \otimes \mathbb{V}$ is well defined, so the action of q on gradient fields, $q(\operatorname{grad} w)$, is well defined provided $\operatorname{grad} w \in \mathring{H}^1 \otimes \mathbb{V}$. But in fact, it is also well defined if $\operatorname{grad} w$ is just in $L_2 \otimes \mathbb{V}$, as stated next, where similar other extensions are also collected.

Lemma 5.2. *Let $q \in H^{-1}(\mathrm{div})$, $\sigma \in H_{\mathrm{dd}}$, $g \in H_{\mathrm{cc}}$, and $\tau \in H_{\mathrm{cd}}$. Then $q \circ \operatorname{grad}$, $\sigma \circ \operatorname{def}$, extends to continuous linear maps such that*

$$q \circ \operatorname{grad} : \mathring{H}^1 \to \mathbb{R}, \qquad (q \circ \operatorname{grad})(w) = -(\operatorname{div} q)(w), \qquad w \in \mathring{H}^1, \tag{83a}$$

$$\sigma \circ \operatorname{def} : \mathring{H}^1 \otimes \mathbb{V} \to \mathbb{R}, \qquad (\sigma \circ \operatorname{def})(u) = -(\operatorname{div} \sigma)(u), \qquad u \in \mathring{H}^1 \otimes \mathbb{V}, \tag{83b}$$

$$\sigma \circ \operatorname{hess} : \mathring{H}^1 \otimes \mathbb{S} \to \mathbb{R}, \qquad (\sigma \circ \operatorname{hess})(w) = (\operatorname{div} \operatorname{div} \sigma)(w), \qquad w \in \mathring{H}^1, \tag{83c}$$

$$g \circ \operatorname{sym} \operatorname{curl} : \mathring{H}^1 \otimes \mathbb{T} \to \mathbb{R}, \qquad (g \circ \operatorname{sym} \operatorname{curl})(\eta) = (\operatorname{curl} g)(\eta), \qquad \eta \in \mathring{H}^1 \otimes \mathbb{T}, \tag{83d}$$

$$g \circ \operatorname{inc} : \mathring{H}^1 \otimes \mathbb{S} \to \mathbb{R}, \qquad (g \circ \operatorname{inc})(\gamma) = (\operatorname{inc} g)(\gamma), \qquad \gamma \in \mathring{H}^1 \otimes \mathbb{S}, \tag{83e}$$

$$\tau \circ \operatorname{curl} : \mathring{H}^1 \otimes \mathbb{T} \to \mathbb{R}, \qquad (\tau \circ \operatorname{curl})(\gamma) = (\operatorname{sym} \operatorname{curl} \tau)(\gamma), \qquad \gamma \in \mathring{H}^1 \otimes \mathbb{S}, \tag{83f}$$

$$\tau \circ \operatorname{dev} \operatorname{grad} : \mathring{H}^1 \otimes \mathbb{V} \to \mathbb{R}, \qquad (\tau \circ \operatorname{dev} \operatorname{grad})(u) = -(\operatorname{div} \tau)(u), \qquad u \in \mathring{H}^1 \otimes \mathbb{V}, \tag{83g}$$

$$\tau \circ \operatorname{curl} \operatorname{def} : \mathring{H}^1 \otimes \mathbb{V} \to \mathbb{R}, \qquad (\tau \circ \operatorname{curl} \operatorname{def})(u) = \frac{1}{2}(\operatorname{curl} \operatorname{div} \tau)(u), \qquad u \in \mathring{H}^1 \otimes \mathbb{V}, \tag{83h}$$

where, component wise, the right hand sides are all duality pairings between H^{-1} and $\mathring{H}^1$.

Proof. For any $\varphi \in \mathcal{D}(\Omega)$, by the definition of the distributional divergence, $(\operatorname{div} q)(\varphi) = -q(\operatorname{grad} \varphi) \equiv -(q \circ \operatorname{grad})(\varphi)$. Since $\operatorname{div} q$ is in $H^{-1}(\mathrm{div})$,

$$|(q \circ \operatorname{grad})(\varphi)| \le \| \operatorname{div} q\|_{H^{-1}} \|\varphi\|_{\mathring{H}^1}$$

showing that $q \circ \operatorname{grad}$ can be continuously extended to the closure of $\mathcal{D}(\Omega)$ in the $\mathring{H}^1$-norm, i.e., to $\mathring{H}^1$. This proves the first statement. The remaining statements are proved similarly, noting that at times we must use the identities of Lemma 2.1: e.g., for any given $\varphi \in \mathcal{D}(\Omega) \otimes \mathbb{V}$, using (28) and (31), we have $(\operatorname{curl} \operatorname{div} \tau)(\varphi) = (\operatorname{div} \top \operatorname{curl} \top \tau)(\varphi) = \top \tau(\operatorname{curl} \top \operatorname{grad} \varphi) = \top \tau(\top \operatorname{dev} \operatorname{grad} \operatorname{curl} \varphi) = 2\tau(\operatorname{curl} \operatorname{def} \varphi)$. □

It was proved in [21] that the dual of $\mathring{H}(\mathrm{div})$ equals $H^{-1}(\mathrm{curl})$, both algebraically and topologically. The next lemma states this together with closely related identities.

Lemma 5.3. *The following equalities of spaces hold algebraically and topologically.*

$$\mathring{H}(\mathrm{div})^* = H^{-1}(\mathrm{curl}), \qquad \mathring{H}(\mathrm{curl})^* = H^{-1}(\mathrm{div}),$$

$$H(\mathrm{div})^* = \widetilde{H}^{-1}(\mathrm{curl}), \qquad H(\mathrm{curl})^* = \widetilde{H}^{-1}(\mathrm{div}).$$

Proof. The first identity was proved in [21, Theorem 2.2]. Here we prove the second and the last (since the third is similar). The proofs are presented step by step below, but a unified strategy for proving all identities will be evident: we use the Riesz representative of a functional in X^* to show that $X^* \hookrightarrow Y$, and then use a regular decomposition of X to prove that $Y \hookrightarrow X^*$.

Step 1. $\underline{\mathring{H}(\operatorname{curl})^* \hookrightarrow H^{-1}(\operatorname{div})}$: To prove that $\mathring{H}(\operatorname{curl})^*$ is continuously embedded in $H^{-1}(\operatorname{div})$, consider an $f \in \mathring{H}(\operatorname{curl})^*$. By the Riesz representation theorem, there is a $u_f \in \mathring{H}(\operatorname{curl})$ such that

$$f(v) = (u_f, v) + (\operatorname{curl} u_f, \operatorname{curl} v), \qquad v \in \mathring{H}(\operatorname{curl}).$$

Choosing $v \in \mathcal{D}(\Omega) \otimes \mathbb{V}$ we find that the equality

$$f = u_f + \operatorname{curl}\operatorname{curl} u_f$$

holds as distributions on Ω. It immediately follows that

$$\begin{aligned} \|f\|^2_{H^{-1}(\operatorname{div})} &= \|u_f + \operatorname{curl}\operatorname{curl} u_f\|^2_{H^{-1}} + \|\operatorname{div} u_f\|^2_{H^{-1}} \\ &\lesssim \|u_f\|_{\mathring{H}(\operatorname{curl})} = \|f\|_{\mathring{H}(\operatorname{curl})^*}. \end{aligned}$$

This proves that the restriction of the distribution f to $\bar{\Omega}$ is an element of $H^{-1}(\operatorname{div})$ and that the embedding of $\mathring{H}(\operatorname{curl})^*$ into $H^{-1}(\operatorname{div})$ is continuous.

Step 2. $\underline{H^{-1}(\operatorname{div}) \hookrightarrow \mathring{H}(\operatorname{curl})^*}$: Let $q \in H^{-1}(\operatorname{div})$. Applying the regular decomposition (44) to split any u in $\mathring{H}(\operatorname{curl})$ as $u = \operatorname{grad} \mathring{S}_{\mathrm{c}}^{(1)} u + \mathring{S}_{\mathrm{c}}^{(0)} u$, we define a functional f_q acting on u by

$$f_q(u) = -(\operatorname{div} q)(\mathring{S}_{\mathrm{c}}^{(1)} u) + q(\mathring{S}_{\mathrm{c}}^{(0)} u). \tag{84}$$

By the continuity of $\mathring{S}_{\mathrm{c}}^{(i)}$,

$$\begin{aligned} |f_q(u)| &\le \|\operatorname{div} q\|_{H^{-1}} \|\mathring{S}_{\mathrm{c}}^{(1)} u\|_{\mathring{H}^1} + \|q\|_{H^{-1}} \|\mathring{S}_{\mathrm{c}}^{(0)} u\|_{\mathring{H}^1} \\ &\lesssim \|q\|_{H^{-1}(\operatorname{div})} \|u\|_{\mathring{H}(\operatorname{curl})}, \end{aligned}$$

so $f_q \in \mathring{H}(\operatorname{curl})^*$ and

$$\|f_q\|_{\mathring{H}(\operatorname{curl})^*} \lesssim \|q\|_{H^{-1}(\operatorname{div})}. \tag{85}$$

Note that f_q is a distribution (which can be seen for instance by the previously proved imbedding showing that f_q is in $H^{-1}(\operatorname{div})$). We now show that the distribution f_q is identical to q. Indeed, by (83a) of Lemma 5.2, for any $\varphi \in \mathcal{D}(\Omega) \times \mathbb{V}$, the definition (84) implies $f_q(\varphi) = q(\operatorname{grad} \mathring{S}_{\mathrm{c}}^{(1)} \varphi + \mathring{S}_{\mathrm{c}}^{(0)} \varphi) = q(\varphi)$, i.e., $f_q = q$. Thus (85) implies that $\|f_q\|_{\mathring{H}(\operatorname{curl})^*} \equiv \|q\|_{\mathring{H}(\operatorname{curl})^*} \lesssim \|q\|_{H^{-1}(\operatorname{div})}$ showing the stated continuous embedding.

Step 3. $\underline{H(\operatorname{curl})^* \hookrightarrow \tilde{\mathring{H}}^{-1}(\operatorname{div})}$: Writing $H(\operatorname{curl})$ as $H(\operatorname{curl}, \Omega)$ to explicitly indicate the domain, we identify any given $f \in H(\operatorname{curl}, \Omega)^*$ with the following distribution in $\mathbb{R}^3$,

$$\tilde{f} \in \mathcal{D}(\mathbb{R}^3)' : \quad \tilde{f}(\varphi) = f(\varphi|_\Omega), \quad \varphi \in \mathcal{D}(\mathbb{R}^3).$$

By the Riesz representation theorem, there is a unique $u_f \in H(\operatorname{curl}, \Omega)$ such that $f(v) = (\operatorname{curl} u_f, \operatorname{curl} v) + (u_f, v)$ for all v in $H(\operatorname{curl}, \Omega)$. Let $\tilde{c}_f$ and $\tilde{u}_f$ denote the extensions by zero of $L_2(\Omega)$-functions $\operatorname{curl} u_f$ and u_f by zero to all $\mathbb{R}^3$. Then, for any $\varphi \in \mathcal{D}(\mathbb{R}^3)$,

$$\tilde{f}(\varphi) = f(\varphi|_\Omega) = (\tilde{c}_f, \operatorname{curl} \varphi)_{L_2(\mathbb{R}^3)} + (\tilde{u}_f, \varphi)_{L_2(\mathbb{R}^3)}.$$

This shows that the following identities hold in $\mathcal{D}(\mathbb{R}^3)$:

$$\tilde{f} = \operatorname{curl} \tilde{c}_f + \tilde{u}_f, \qquad \operatorname{div} \tilde{f} = \operatorname{div} \tilde{u}_f.$$

These two identities give these corresponding bounds

$$\begin{aligned}\|f\|_{\mathring{\tilde{H}}^{-1}} &= \|\tilde{f}\|_{H^{-1}(\mathbb{R}^3)} \le \|\operatorname{curl}\tilde{c}_f\|_{H^{-1}(\mathbb{R}^3)} + \|\tilde{u}_f\|_{H^{-1}(\mathbb{R}^3)} \\ &\lesssim \|\tilde{c}_f\|_{L_2(\mathbb{R}^3)} + \|\tilde{u}_f\|_{L_2(\mathbb{R}^3)} \lesssim \|u_f\|_{H(\text{curl})}, \\ \|\operatorname{div} f\|_{\mathring{\tilde{H}}^{-1}} &= \|\operatorname{div}\tilde{f}\|_{H^{-1}(\mathbb{R}^3)} = \|\operatorname{div}\tilde{u}_f\|_{H^{-1}(\mathbb{R}^3)} \lesssim \|\tilde{u}_f\|_{L^2(\mathbb{R}^3)} = \|u_f\|_{L_2},\end{aligned}$$

which prove that $\|f\|_{\mathring{\tilde{H}}^{-1}(\text{div})} \lesssim \|u_f\|_{H(\text{curl})} = \|f\|_{H(\text{curl})^*}$ using the Riesz isometry.

Step 4. $\underline{\mathring{\tilde{H}}^{-1}(\text{div}) \hookrightarrow H(\text{curl})^*}$: In exactly the same way the regularized Bogovskiĭ operators of (43) give the regular decomposition (44), the regularized Poincaré operators of [16] show that there are continuous operators $S_{\text{c}}^{(0)} : H(\text{curl}) \to H^1 \otimes \mathbb{V}$ and $S_{\text{c}}^{(1)} : H(\text{curl}) \to H^1$ such that any $u \in H(\text{curl})$ can be decomposed into

$$u = S_{\text{c}}^{(0)}\, u + \operatorname{grad} S_{\text{c}}^{(1)}\, u. \tag{86}$$

Let $q \in \mathring{\tilde{H}}^{-1}(\text{div})$. Using the regular decomposition, we define a functional

$$f_q(u) = -(\operatorname{div} q)(S_{\text{c}}^{(1)}\, u) + q(S_{\text{c}}^{(u)})$$

where, on the right hand side, the functional actions are duality pairings between $\mathring{\tilde{H}}^{-1}$ and H^1, well defined in view of (20). Hence

$$|f_q(u)| \le \|\operatorname{div} q\|_{\mathring{\tilde{H}}^{-1}} \|\, S_{\text{c}}^{(1)}\, u\|_{H^1} + \|q\|_{\mathring{\tilde{H}}^{-1}} \|\, S_{\text{c}}^{(0)}\, u\|_{H^1} \lesssim \|q\|_{\mathring{\tilde{H}}^{-1}(\text{div})} \|u\|_{H(\text{curl})}$$

Therefore f_q is in $H(\text{curl})^*$ and

$$\|f_q\|_{H(\text{curl})^*} \lesssim \|q\|_{\mathring{\tilde{H}}^{-1}(\text{div})}. \tag{87}$$

Since we have already shown that $H(\text{curl})^*$ is embedded into the subspace of $\mathbb{R}^3$-distributions in $\mathring{\tilde{H}}^{-1}(\text{div})$, we conclude that $f_q \in \mathcal{D}(\mathbb{R}^3)'$ satisfies, for any $\varphi \in \mathcal{D}(\mathbb{R}^3)$,

$$f_q(\varphi) = -(\operatorname{div} q)(S_{\text{c}}^{(1)}\, \varphi) + q(S_{\text{c}}^{(0)}\, \varphi) = q(\operatorname{grad} S_{\text{c}}^{(1)}\, \varphi + S_{\text{c}}^{(0)}\, \varphi) = q(\varphi),$$

i.e., f_q and q coincide as distributions. Combined with (87), we have thus shown that $\|q\|_{H(\text{curl})^*} = \|f_q\|_{H(\text{curl})^*} \lesssim \|q\|_{\mathring{\tilde{H}}^{-1}(\text{div})}$.

Steps 1 and 2 prove the second identity of the lemma. Steps 3 and 4 prove the last identity of the lemma. The proofs of the remaining identities are similar. ☐

In the next theorem we show that more dualities such as those in Lemma 5.3 can be read off the diagrams with and without boundary conditions. Let $\mathcal{L} : H^1 \to \mathring{\tilde{H}}^{-1} \equiv (H^1)^*$ denote the Riesz map of H^1 defined by

$$(\mathcal{L}w)(v) = (w, v)_{H^1} \equiv (\operatorname{grad} w, \operatorname{grad} v) + (w, v), \quad w, v \in \mathring{H}^1, \tag{88}$$

where $(\cdot,\cdot)_X$ denotes the inner product of X, and when the subscript is absent, we interpret it as L_2 products, as described previously in (8). By the Riesz representation theorem, recalling that our spaces are over the real field, $\mathcal{L}$ is a linear invertible isometry, so for any $q, r \in \mathring{\tilde{H}}^{-1}$,

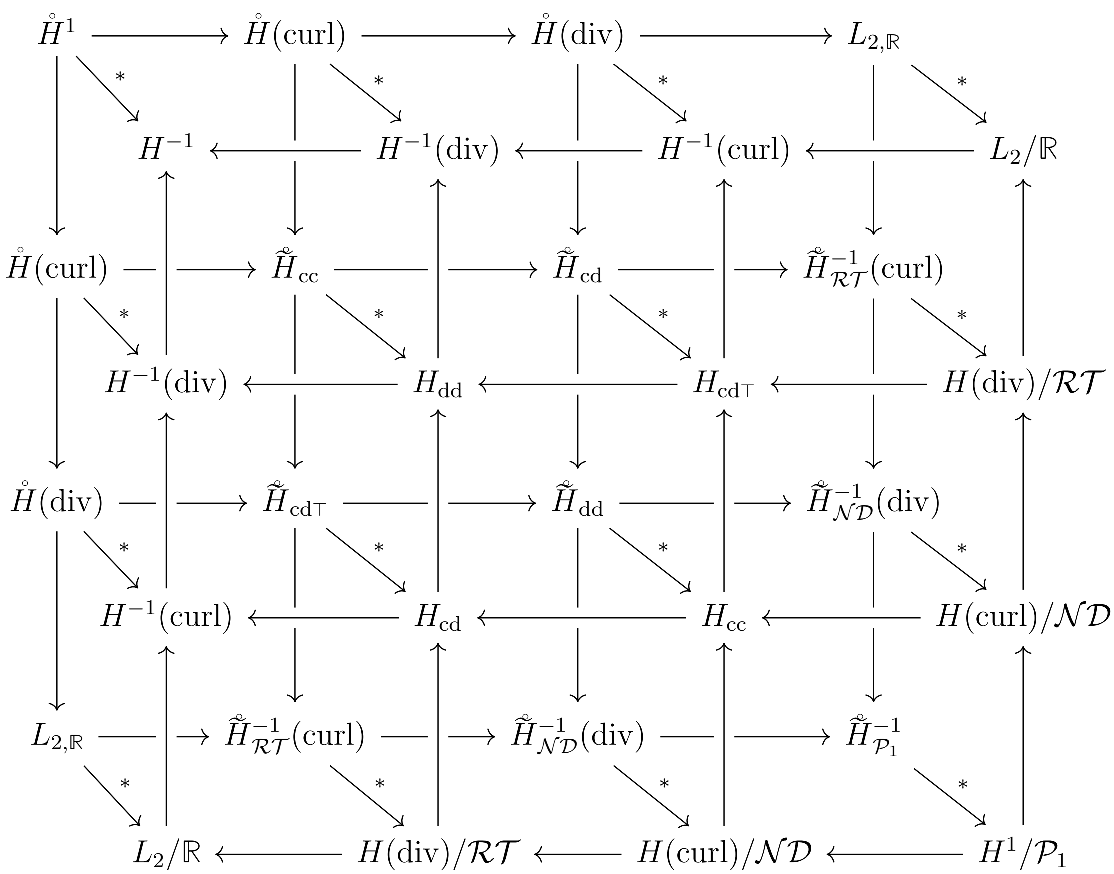
$$(q, r)_{\mathring{\tilde{H}}^{-1}} = (\mathcal{L}^{-1}q, \mathcal{L}^{-1}r)_{H^1} = r(\mathcal{L}^{-1}q),$$

where we have used (88) in the last step. In particular, if a $v \in L_2 \subset H^{-1}$ is used in place of r above, the functional action becomes an L_2 product and we obtain

$$(q, v)_{\mathring{\tilde{H}}^{-1}} = (\mathcal{L}^{-1}q, v), \qquad v \in L_2. \tag{89}$$

Note that $\mathcal{L}^{-1}q$ is the result of solving a Neumann problem with q as the source. When q is a vector or matrix field, by $\mathcal{L}^{-1}q$ we mean the respective vector or matrix field obtained by component-wise application of $\mathcal{L}^{-1}$.

Theorem 5.4. *The diagrams* (25) *and* (82) *are in duality in the sense displayed below, where the second diagram has been rearranged to easily display the duality* ($*$) *correspondences.*



(90)

Proof. The strategy of this proof is the same as what was outlined in the beginning of the proof of Lemma 5.3. We now focus on the spaces of matrix-valued functions.

Step 1. $\underline{\tilde{\mathring{H}}^*_{\mathrm{cc}} \hookrightarrow H_{\mathrm{dd}}}$: Let $f \in \tilde{\mathring{H}}^*_{\mathrm{cc}}$. Then, by Riesz representation, there is a $g_f \in \tilde{\mathring{H}}_{\mathrm{cc}}$ such that $f(g) = (g_f, g)_{\tilde{\mathring{H}}_{\mathrm{cc}}}$ for all $g \in \tilde{\mathring{H}}_{\mathrm{cc}}$. Expanding out the $\tilde{\mathring{H}}_{\mathrm{cc}}$ inner product and using (89), for any $\varphi \in \mathcal{D}(\Omega) \otimes \mathbb{S}$,

$$
\begin{aligned}
f(\varphi) &= (g_f, \varphi)_{H^{-1}} + (\operatorname{curl} g_f, \operatorname{curl} \varphi)_{H^{-1}} + (\operatorname{inc} g_f, \operatorname{inc} \varphi)_{H^{-1}} \\
&= (\mathcal{L}^{-1} g_f, \varphi) + (\mathcal{L}^{-1} \operatorname{curl} g_f, \operatorname{curl} \varphi) + (\mathcal{L}^{-1} \operatorname{inc} g_f, \operatorname{inc} \varphi).
\end{aligned}
$$

Thus

$$
f = \mathcal{L}^{-1} g_f + \operatorname{curl} \mathcal{L}^{-1} \operatorname{curl} g_f + \operatorname{inc} \mathcal{L}^{-1} \operatorname{inc} g_f, \tag{91}
$$

a sum of three terms, which are in H^1, L_2, and H^{-1}, respectively. Since $H^1 \hookrightarrow L_2 \hookrightarrow H^{-1}$,

$$\begin{aligned}
\|f\|_{H^{-1}} &\le \|\mathcal{L}^{-1} g_f\|_{H^{-1}} + \|\operatorname{curl} \mathcal{L}^{-1} \operatorname{curl} g_f\|_{H^{-1}} + \|\operatorname{inc} \mathcal{L}^{-1} \operatorname{inc} g_f\|_{H^{-1}} \\
&\lesssim \|\mathcal{L}^{-1} g_f\|_{H^1} + \|\operatorname{curl} \mathcal{L}^{-1} \operatorname{curl} g_f\|_{L_2} + \|\operatorname{inc} \mathcal{L}^{-1} \operatorname{inc} g_f\|_{H^{-1}} \\
&\lesssim \|\mathcal{L}^{-1} g_f\|_{H^1} + \|\mathcal{L}^{-1} \operatorname{curl} g_f\|_{H^1} + \|\mathcal{L}^{-1} \operatorname{inc} g_f\|_{H^1} \\
&= \|g_f\|_{\mathring{\widetilde{H}}^{-1}} + \|\operatorname{curl} g_f\|_{\mathring{\widetilde{H}}^{-1}} + \|\operatorname{inc} g_f\|_{\mathring{\widetilde{H}}^{-1}} \\
&\lesssim \|g_f\|_{\mathring{\widetilde{H}}_{\mathrm{cc}}} = \|f\|_{\mathring{\widetilde{H}}^*_{\mathrm{cc}}}
\end{aligned}$$

where we have used the Riesz isometry multiple times as well as the continuity of the derivative $\partial_i : H^m \to H^{m-1}$ for some integers m (see e.g., [24, Theorem 1.4.4.6]). From (91) we also see that $\operatorname{div} f = \operatorname{div} \mathcal{L}^{-1} g_f$ and $\operatorname{div}\operatorname{div} f = \operatorname{div}\operatorname{div} \mathcal{L}^{-1} g_f$, so f is indeed in H_{dd}. Moreover,

$$\begin{aligned}
\|f\|^2_{H_{\mathrm{dd}}} &= \|f\|^2_{H^{-1}} + \|\operatorname{div} \mathcal{L}^{-1} g_f\|^2_{H^{-1}} + \|\operatorname{div}\operatorname{div} \mathcal{L}^{-1} g_f\|^2_{H^{-1}} \\
&\lesssim \|f\|^2_{H^{-1}} + \|\mathcal{L}^{-1} g_f\|^2_{H^1} = \|f\|^2_{H^{-1}} + \|g_f\|^2_{\mathring{\widetilde{H}}^{-1}} \\
&\lesssim \|f\|_{\mathring{\widetilde{H}}^*_{\mathrm{cc}}},
\end{aligned}$$

thus completing the proof of continuity of the embedding of f into H_{dd}.

Step 2. $\underline{H_{\mathrm{dd}} \hookrightarrow \mathring{\widetilde{H}}^*_{\mathrm{cc}}}$: Let $\sigma \in H_{\mathrm{dd}}$. Decomposing a $g \in \mathring{\widetilde{H}}_{\mathrm{cc}}$ using (45) as $g = \mathring{S}^{(0)}_{\mathrm{cc}}\, g + \operatorname{def} \mathring{S}^{(1)}_{\mathrm{cc}}\, g + \operatorname{hess} \mathring{S}^{(2)}_{\mathrm{cc}}\, g$, we define a linear functional f_σ acting on g as follows:

$$f_\sigma(g) = \sigma(\mathring{S}^{(0)}_{\mathrm{cc}}\, g) - (\operatorname{div}\sigma)(\mathring{S}^{(1)}_{\mathrm{cc}}\, g) + (\operatorname{div}\operatorname{div}\sigma)(\mathring{S}^{(2)}_{\mathrm{cc}}\, g).$$

By the continuity of $\mathring{S}^{(i)}_{\mathrm{cc}}$ (see Theorem 3.4), $|f_\sigma(g)| \lesssim \|\sigma\|_{H_{\mathrm{dd}}} \|g\|_{\mathring{\widetilde{H}}_{\mathrm{cc}}}$ so f_σ is in $\mathring{\widetilde{H}}^*_{\mathrm{cc}}$ and

$$\|f_\sigma\|_{\mathring{\widetilde{H}}^*_{\mathrm{cc}}} \lesssim \|\sigma\|_{H_{\mathrm{dd}}}. \tag{92}$$

By the previous embedding, we know that f_σ is in H_{dd} (in particular in $H^{-1} \otimes \mathbb{S}$) and is therefore a distribution on Ω. We claim that f_σ and σ are identical distributions. Indeed, for any $\varphi \in \mathcal{D}(\Omega) \otimes \mathbb{S}$,

$$\begin{aligned}
f_\sigma(\varphi) &= \sigma(\mathring{S}^{(0)}_{\mathrm{cc}}\, \varphi) - (\operatorname{div}\sigma)(\mathring{S}^{(1)}_{\mathrm{cc}}\, \varphi) + (\operatorname{div}\operatorname{div}\sigma)(\mathring{S}^{(2)}_{\mathrm{cc}}\, \varphi) \\
&= \sigma(\mathring{S}^{(0)}_{\mathrm{cc}}\, \varphi + \operatorname{def} \mathring{S}^{(1)}_{\mathrm{cc}}\, \varphi + \operatorname{hess} \mathring{S}^{(2)}_{\mathrm{cc}}\, \varphi) = \sigma(\varphi)
\end{aligned}$$

where we have used (83b) and (83c) of Lemma 5.2. Combined with (92), we have $\|\sigma\|_{\mathring{\widetilde{H}}^*_{\mathrm{cc}}} \equiv \|f_\sigma\|_{\mathring{\widetilde{H}}^*_{\mathrm{cc}}} \lesssim \|\sigma\|_{H_{\mathrm{dd}}}$, thus completing the proof of continuity of the embedding of σ into $\mathring{\widetilde{H}}^*_{\mathrm{dd}}$.

Step 3. $\underline{\mathring{\widetilde{H}}^*_{\mathrm{cd}} \hookrightarrow H_{\mathrm{cd}\top}}$: Let $f \in \mathring{\widetilde{H}}^*_{\mathrm{cd}}$. By the Riesz representation theorem, there is a unique $\tau_f \in \mathring{\widetilde{H}}_{\mathrm{cd}}$ satisfying $f(\tau) = (\tau_f, \tau)_{\mathring{\widetilde{H}}_{\mathrm{cd}}}$ for all $\tau \in \mathring{\widetilde{H}}_{\mathrm{cd}}$. Choosing $\tau = \varphi \in \mathcal{D} \otimes \mathbb{T}$, the definition of $\mathring{\widetilde{H}}_{\mathrm{cd}}$-inner product and (89) imply

$$\begin{aligned}
f(\varphi) = {} & (\mathcal{L}^{-1}\tau_f, \varphi) + (\mathcal{L}^{-1} \operatorname{div}\tau_f, \operatorname{div}\varphi) + (\mathcal{L}^{-1} \operatorname{sym}\operatorname{curl} \top \tau_f, \operatorname{sym}\operatorname{curl}\varphi) \\
& + (\mathcal{L}^{-1} \operatorname{curl}\operatorname{div}\tau_f, \operatorname{curl}\operatorname{div}\varphi).
\end{aligned}$$

Hence

$$\begin{aligned}
f = {} & \mathcal{L}^{-1}\tau_f - \operatorname{grad} \mathcal{L}^{-1} \operatorname{div}\tau_f + \top \operatorname{curl}\operatorname{sym} \mathcal{L}^{-1} \operatorname{sym}\operatorname{curl} \top \tau_f \\
& - \operatorname{grad}\operatorname{curl} \mathcal{L}^{-1} \operatorname{curl}\operatorname{div}\tau_f.
\end{aligned} \tag{93}$$

Applying $\operatorname{div} \top$, the identity (31) shows that last term vanishes and

$$\operatorname{div} \top f = \operatorname{div} \top \mathcal{L}^{-1}\tau_f - \operatorname{div} \top \operatorname{grad} \mathcal{L}^{-1} \operatorname{div}\tau_f. \tag{94}$$

Moreover, applying $\operatorname{curl}\operatorname{div}\top$ to both sides of (93) and using the identity (28),

$$\operatorname{curl}\operatorname{div}\top f = \operatorname{curl}\operatorname{div}\top \mathcal{L}^{-1}\tau_f. \tag{95}$$

Also,

$$\operatorname{sym}\operatorname{curl} f = \operatorname{sym}\operatorname{curl}\mathcal{L}^{-1}\tau_f + \operatorname{inc}\operatorname{sym}\mathcal{L}^{-1}\operatorname{sym}\operatorname{curl}\top\tau_f. \tag{96}$$

Equations (93), (94), (95) and (96) imply that

$$\|f\|_{H_{\mathrm{cd}\top}} \lesssim \|\tau_f\|_{\mathring{\tilde{H}}_{\mathrm{cd}}} = \|f\|_{\mathring{\tilde{H}}^*_{\mathrm{cd}}}$$

where we have used the isometry of $\mathcal{L}^{-1}$ and $f \mapsto \tau_f$. This proves that $\mathring{\tilde{H}}^*_{\mathrm{cd}} \hookrightarrow H_{\mathrm{cd}\top}$.

Step 4. $\underline{H_{\mathrm{cd}\top} \hookrightarrow \mathring{\tilde{H}}^*_{\mathrm{cd}}}$: Let $\tau \in H_{\mathrm{cd}\top}$. Define a linear functional f_τ acting on any η in $\mathring{\tilde{H}}_{\mathrm{cd}}$ as follows:

$$f_\tau(\eta) = \tau(\mathring{S}^{(0)}_{\mathrm{cd}}\,\eta) + (\operatorname{sym}\operatorname{curl}\tau)(\mathring{S}^{(1)}_{\mathrm{cd}}\,\eta) - (\operatorname{div}\top\tau)(\mathring{S}^{(3)}_{\mathrm{cd}}\,\eta) - (\frac{1}{2}\operatorname{curl}\operatorname{div}\top\tau)(\mathring{S}^{(2)}_{\mathrm{cd}}\,\eta).$$

All the terms on the right are well defined duality pairings since $\tau, \operatorname{sym}\operatorname{curl}\tau, \operatorname{div}\top\tau$, and $\operatorname{curl}\operatorname{div}\top\tau$, have $\tilde{H}^{-1}$ components for any $\tau \in H_{\mathrm{cd}\top}$ and since each $\mathring{S}^{(i)}_{\mathrm{cd}}\,\eta$ has $\mathring{H}^1$ components per Theorem 3.10. The continuity of $\mathring{S}^{(i)}_{\mathrm{cd}}$ asserted by the same theorem gives

$$|f_\tau(\eta)| \lesssim \|\tau\|_{H_{\mathrm{cd}\top}}\|\eta\|_{\mathring{\tilde{H}}_{\mathrm{cd}}}. \tag{97}$$

Next, observe that for any $\varphi \in \mathcal{D}(\Omega)\otimes\mathbb{T}$, decomposing φ by Theorem 3.10 into $\varphi = \mathring{S}^{(0)}_{\mathrm{cd}}\,\varphi + \operatorname{curl}\mathring{S}^{(1)}_{\mathrm{cd}}\,\varphi + \top\operatorname{dev}\operatorname{grad}\mathring{S}^{(2)}_{\mathrm{cd}}\,\varphi + \operatorname{curl}\operatorname{def}\mathring{S}^{(3)}_{\mathrm{cd}}\,\varphi$, and using (83f), (83g) and (83h) of Lemma 5.2,

$$\begin{aligned} f_\tau(\varphi) &= \tau(\mathring{S}^{(0)}_{\mathrm{cd}}\,\varphi) + \tau(\operatorname{curl}\mathring{S}^{(1)}_{\mathrm{cd}}\,\varphi) + \tau(\top\operatorname{dev}\operatorname{grad}\mathring{S}^{(3)}_{\mathrm{cd}}\,\varphi) + \tau(\operatorname{curl}\operatorname{def}\mathring{S}^{(2)}_{\mathrm{cd}}\,\varphi) \\ &= \tau(\varphi). \end{aligned}$$

Hence f_τ and τ are the same distribution. Using (97), $\|\tau\|_{\mathring{\tilde{H}}^*_{\mathrm{cd}}} \equiv \|f_\tau\|_{\mathring{\tilde{H}}^*_{\mathrm{cd}}} \lesssim \|\tau\|_{H_{\mathrm{cd}\top}}$, which proves that $H_{\mathrm{cd}\top} \hookrightarrow \mathring{\tilde{H}}^*_{\mathrm{cd}}$ is a continuous embedding.

Step 5. $\underline{\mathring{\tilde{H}}^*_{\mathrm{dd}} \hookrightarrow H_{\mathrm{cc}}}$: Denoting the Riesz representative of any $f \in \mathring{\tilde{H}}^*_{\mathrm{dd}}$ by $\sigma_f \in \mathring{\tilde{H}}_{\mathrm{dd}}$, its defining equation $f(\sigma) = (\sigma_f, \sigma)_{\mathring{\tilde{H}}_{\mathrm{dd}}}$ for all $\sigma \in \mathring{\tilde{H}}_{\mathrm{dd}}$ gives a formula for f as in the previous cases:

$$f = \mathcal{L}^{-1}\sigma_f - \operatorname{grad}\mathcal{L}^{-1}\operatorname{div}\sigma_f + \operatorname{hess}\mathcal{L}^{-1}\operatorname{div}\operatorname{div}\sigma_f.$$

Then $\operatorname{curl} f = \operatorname{curl}\mathcal{L}^{-1}\sigma_f$ is in $L_2\otimes\mathbb{V}$ and $\operatorname{inc} f = \operatorname{inc}\mathcal{L}^{-1}\sigma_f$ is in $\tilde{H}^{-1}\otimes\mathbb{S}$. Hence f is in H_{cc} and by Riesz isometries, $\|f\|_{H_{\mathrm{cc}}} \lesssim \|f\|_{\mathring{\tilde{H}}^*_{\mathrm{dd}}}$.

Step 6. $\underline{H_{\mathrm{cc}} \hookrightarrow \mathring{\tilde{H}}^*_{\mathrm{dd}}}$: Let $g \in H_{\mathrm{cc}}$. We will show that g can be identified with the functional f_g acting on $\sigma \in \mathring{\tilde{H}}_{\mathrm{dd}}$ by

$$f_g(\sigma) = g(\mathring{S}^{(0)}_{\mathrm{dd}}\,\sigma) + (\operatorname{curl} g)(\mathring{S}^{(1)}_{\mathrm{dd}}\,\sigma) + (\operatorname{inc} g)(\mathring{S}^{(2)}_{\mathrm{dd}}).$$

By the regular decomposition result of Theorem 3.7, this implies that

$$|f_g(\sigma)| \lesssim \|g\|_{H_{\mathrm{cc}}}\|\sigma\|_{\mathring{\tilde{H}}_{\mathrm{dd}}} \tag{98}$$

and also using (83d) and (83e) of Lemma 5.2, we find that

$$f_g(\varphi) = g(\mathring{S}^{(0)}_{\mathrm{dd}}\,\varphi + \operatorname{sym}\operatorname{curl}\mathring{S}^{(1)}_{\mathrm{dd}}\,\varphi + \operatorname{inc}\mathring{S}^{(2)}_{\mathrm{dd}}\,\varphi) = g(\varphi)$$

for any $\varphi \in \mathcal{D}(\Omega)\otimes\mathbb{S}$, i.e., f_g and g are the same distribution. Hence (98) shows that $\|g\|_{\mathring{\tilde{H}}^*_{\mathrm{dd}}} = \|f_g\|_{\mathring{\tilde{H}}^*_{\mathrm{dd}}} \lesssim \|g\|_{H_{\mathrm{cc}}}$ thus establishing the continuous embedding of H_{cc} into $\mathring{\tilde{H}}^*_{\mathrm{dd}}$.

To conclude the proof, note that Steps 1 and 2 prove that $\mathring{\widetilde{H}}^*_{\mathrm{cc}} = H_{\mathrm{dd}}$. Steps 3 and 4 prove that $\mathring{\widetilde{H}}^*_{\mathrm{cd}} = H_{\mathrm{cd}\top}$ and $\mathring{\widetilde{H}}^*_{\mathrm{cd}\top} = H_{\mathrm{cd}}$. Steps 5 and 6 prove that $\mathring{\widetilde{H}}^*_{\mathrm{dd}} = H_{\mathrm{cc}}$. To establish the remaining nontrivial duality identities in (90), it suffices to use the identities of Lemma 5.3 and the fact that for any Hilbert space X and its closed subspace Z, the dual of the quotient space $(X/Z)^*$ is isomorphic to the annihilator $Z^\perp = \{x' \in X' : x'(z) = 0 \text{ for all } z \in Z\}$ contained in X'. For example, with $X = H(\mathrm{curl})$ and $Z = \mathcal{ND}$, we find that $(H(\mathrm{curl})/\mathcal{ND})^*$ is isomorphic to the annihilator of $\mathcal{ND}^\perp$ contained in $H(\mathrm{curl})^* = \mathring{\widetilde{H}}^{-1}(\mathrm{div})$, which is exactly the same as $\mathring{\widetilde{H}}^{-1}_{\mathcal{ND}}(\mathrm{div})$, i.e., $\big(H(\mathrm{curl})/\mathcal{ND}\big)^* = \mathring{\widetilde{H}}^{-1}_{\mathcal{ND}}(\mathrm{div})$. Taking duals on both sides we obtain, due to the reflexivity of Hilbert spaces, $\mathring{\widetilde{H}}^{-1}_{\mathcal{ND}}(\mathrm{div})^* = H(\mathrm{curl})/\mathcal{ND}$, which is one of the identities indicated in (90). □

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