HW 2 Due: 29 April 2026

As always, your answer will be graded on the quality of presentation as well as the correct answer. To get a good score: write your answer neatly, use complete sentences, and *justify your work*.

1. Suppose that R is an integral domain and M is a left R-module. Let's write

$$U = R \setminus \{0\} \qquad \text{and} \qquad \operatorname{Tor}(M) = \{m \in M \mid \text{there exists } r \in U \text{ such that } rm = 0\}.$$

We showed that $U^{-1}M$ and $\operatorname{Frac}(R) \otimes_R M$ are isomorphic *as tensor products*. (See [Ash10, Section 8.5].)

(a) Prove that for all $d \in U$ and $m \in M$:

$$\frac{1}{d} \otimes_R m = 0 \qquad \text{if and only if} \qquad m \in \operatorname{Tor}(M).$$

(b) Let $\beta\colon \operatorname{Frac}(R) \times M \to \operatorname{Frac}(R) \otimes_R M$ be the bilinear map with which $\operatorname{Frac}(R) \otimes_R M$ is equipped. Then the function

$$\begin{aligned} \iota\colon M &\to \operatorname{Frac}(R) \otimes_R M \\ m &\mapsto \beta(1, m) \end{aligned}$$

is an R-module homomorphism by the definition of bilinear. Prove that

$$\ker(\iota) = \operatorname{Tor}(M).$$

References

[Ash10] Robert B. Ash, *Abstract Algebra: The Basic Graduate Year*, 2010.