HW 4 Due: 18 March 2026

As always, your answer will be graded on the quality of presentation as well as the correct answer. To get a good score: write your answer neatly, use complete sentences, and *justify your work*.

1. Let p be a prime number. For $n \in \mathbb{Z}_{>0}$, let's write $M_n = \mathbb{Z}/p^n\mathbb{Z} \in \operatorname{Obj}(\mathbb{Z}\text{-}\mathsf{Mod})$. For any $m, n \in \mathbb{Z}_{\geq 1}$ with $m \leq n$, define $\mu_{mn} \in \operatorname{Hom}_{\mathbb{Z}\text{-}\mathsf{Mod}}(M_m, M_n)$ by the demand that $\mu_{mn}(1 + p^m\mathbb{Z}) = p^{n-m} + p^n\mathbb{Z}$. We have shown that this is a direct system of $\mathbb{Z}$-modules.

   (a) Prove
   $$\varinjlim M_n = \{q + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z} \mid q + \mathbb{Z} \text{ has order a power of } p\}.$$

   (b) Prove that $\{q + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z} \mid q + \mathbb{Z} \text{ has order a power of } p\}$ is divisible.