HW 3 Due: 4 March 2026

As always, your answer will be graded on the quality of presentation as well as the correct answer. To get a good score: write your answer neatly, use complete sentences, and *justify your work*.

1. For any ring R, let $R\text{-}\mathsf{Mod}$ be the category of left R-modules and let $\mathsf{Mod}\text{-}R$ be the category of right R-modules. Now let $\mathcal{F}\colon R\text{-}\mathsf{Mod} \to \mathbb{Z}\text{-}\mathsf{Mod}$ be the forgetful functor and $\mathcal{H}\colon \mathbb{Z}\text{-}\mathsf{Mod} \to \mathbb{Z}\text{-}\mathsf{Mod}$ be the (contravariant) functor $\operatorname{Hom}_{\mathbb{Z}\text{-}\mathsf{Mod}}(\bullet, \mathbb{Q}/\mathbb{Z})$.
   (a) For any $M \in \operatorname{Obj}(R\text{-}\mathsf{Mod})$, bestow a right R-module structure on $(\mathcal{H} \circ \mathcal{F})(M)$, thereby obtaining a functor $\mathcal{P}\colon R\text{-}\mathsf{Mod} \to \mathsf{Mod}\text{-}R$.
   (b) Suppose that $M, N \in \operatorname{Obj}(R\text{-}\mathsf{Mod})$ and that $\phi \in \operatorname{Hom}_{R\text{-}\mathsf{Mod}}(M, N)$. Prove ϕ is injective if and only if $\mathcal{P}(\phi)$ is surjective.
   (c) Let $M \in \operatorname{Obj}(R\text{-}\mathsf{Mod})$. Use adjunction to prove that M is flat if and only if $\mathcal{P}(M)$ is injective.
2. Suppose that R is an integral domain and I is an ideal of R. Prove

$$R/I \text{ is flat (as an } R\text{-module)} \qquad \text{if and only if} \qquad \text{either } I = \{0\} \text{ or } I = R.$$

References

[Ash10] Robert B. Ash, *Abstract Algebra: The Basic Graduate Year*, 2010.