Exercise 1:
Suppose $\mathbf{C}$ is a category with zero objects. Choose any $A \in \mathrm{Obj}(\mathbf{C})$. Let's write 0_A for the zero morphism in $\mathrm{Hom}_{\mathbf{C}}(A, A)$ and id_A for the identity morphism in $\mathrm{Hom}_{\mathbf{C}}(A, A)$. Prove that A is a zero object if and only if $0_A = \mathrm{id}_A$.

Solution. "$\Rightarrow$"

Suppose A is a zero object of a category $\mathbf{C}$. By definition of the zero object, there is a unique morphism $A \to A$, i.e. $\mathrm{Hom}(A, A) = \{A \to A\}$. This is the zero morphism, as it "factors" through a zero object, i.e., we have the following commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & A \\ \downarrow & \nearrow & \\ A & & \end{array}$$

Where $A \to A$ is the unique arrow in question, $A \to A$ is the unique arrow to the zero object, and $A \to A$ is the unique arrow from the zero object, so the unique arrow in question must be the zero morphism 0_A.

Furthermore, in order to call $\mathbf{C}$ a category (accurately) it is necessary that $\mathrm{id}_A \in \mathrm{Hom}(A, A)$, since there is only one arrow in $\mathrm{Hom}(A, A)$, it must be id_A, and thus by the previous analysis, we have $0_A = \mathrm{id}_A$.

"$\Leftarrow$"

Suppose $A \in \mathrm{Obj}(\mathbf{C})$ and that $0_A = \mathrm{id}_A$. Let Z be a zero object in $\mathbf{C}$, then we have the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{id}_A} & A \\ \downarrow & \nearrow & \\ Z & & \end{array}$$

In words, the unique morphism to the zero object, followed by the unique morphism out of the zero object is equal to the identity on A, this is the meaning of $0_A = \mathrm{id}_A$. But then we also have the following commutative diagram:

$$\begin{array}{ccc} Z & \xrightarrow{\mathrm{id}_Z} & Z \\ \downarrow & \nearrow & \\ A & & \end{array}$$

In words, composition in the reverse order gives the identity on Z, thus the unique morphism to Z and the unique morphism from Z are inverses of one another, and $A \cong Z$.

□

Exercise 2:
Suppose that $\mathbf{C}, \mathbf{D}$ are categories with zero objects, and $F : \mathbf{C} \to \mathbf{D}$ is a functor. Suppose that F has the property that it sends zero morphisms to zero morphisms. Prove that if Z is a zero object in $\mathrm{Obj}(\mathbf{C})$, then $F(Z)$ is a zero object in $\mathrm{Obj}(\mathbf{D})$. (recall that functors must send identity

morphisms to identity morphisms.)

Solution. By exercise 1 and the fact that F maps zero morphisms to zero morphisms, we have the following in the category **D**:

$$0_{F(Z)} = F(0_Z) = F(\mathrm{id}_Z) = \mathrm{id}_{F(Z)},$$

thus applying exercise 1 again shows that $F(Z)$ is a zero object in **D**.

□

Exercise 3:
Suppose R is a commutative ring and U is a multiplicative subset of R such that $1 \in U$ and $0 \notin U$. Prove that $U^{-1}R \otimes_R M$ and $U^{-1}M$ are isomorphic as $U^{-1}R$-modules. (See Ash section 8.5.11 for the definition of $U^{-1}M$.)

Solution. We will show that $U^{-1}M$ satisfies the universal property of the tensor product $U^{-1}R \otimes_R M$ as a $U^{-1}R$-module. Suppose $B : U^{-1}R \times M \to N$ is an R-balanced, $U^{-1}R$-linear-in-the-first-argument, additive-in-the-second-argument mapping. Let $r \in R, u \in U$, and $m \in M$. Define a mapping $A : U^{-1}R \times M \to U^{-1}M$ via

$$A\left(\left(\frac{r}{u}, m\right)\right) = \frac{r \cdot m}{u}.$$

Then A is also an R-balanced, $U^{-1}R$-linear-in-the-first-argument, additive-in-the-second-argument mapping, indeed, for all $r, r' \in R$, $u, u' \in U$, and $m, m' \in M$ we have the following four conditions:

$$A\left(\left(\frac{r'}{u'} \cdot \frac{r}{u}, m\right)\right) = \frac{r'r \cdot m}{u'u} = \frac{r'}{u'} \cdot \frac{rm}{u} = \frac{r'}{u'} A\left(\left(\frac{r}{u}, m\right)\right),$$

$$A\left(\left(\frac{r}{u} + \frac{r'}{u'}, m\right)\right) = \left(\frac{r}{u} + \frac{r'}{u'}\right) m = \frac{rm}{u} + \frac{r'm}{u'} = A\left(\left(\frac{r}{u}, m\right)\right) + A\left(\left(\frac{r'}{u'}, m\right)\right),$$

$$A\left(\left(\frac{r}{u}, m + m'\right)\right) = \frac{r}{u}(m + m') = \frac{rm}{u} + \frac{rm'}{u} = A\left(\left(\frac{r}{u}, m\right)\right) + A\left(\left(\frac{r}{u}, m'\right)\right),$$

and

$$A\left(\left(\frac{r}{u}, r'm\right)\right) = \frac{rr'm}{u} = \frac{rr'}{u} \cdot m = A\left(\left(\frac{rr'}{u}, m\right)\right).$$

Now let $\tilde{B} : U^{-1}M \to N$ be defined as

$$\tilde{B}\left(\frac{m}{u}\right) = B\left(\frac{1}{u}, m\right),$$

then $\tilde{B}$ is $U^{-1}R$-linear, indeed we have

$$\tilde{B}\left(\frac{rm}{u}\right) = B\left(\frac{1}{u}, rm\right) = B\left(\frac{r}{u}, m\right) = \frac{r}{u} B(1, m) = \frac{r}{u}\tilde{B}(m),$$

and

$$\tilde{B}(m + m') = B(1, m = m') = B(1, m) + B(1, m') = \tilde{B}(m) + \tilde{B}(m').$$

We have $\tilde{B} \circ A = B$, indeed

$$\tilde{B}(A(\frac{r}{u}, m)) = \tilde{B}(\frac{rm}{u}) = B(\frac{1}{u}, rm) = B(\frac{r}{u}, m).$$

So we have the following commutative diagram:

$$\begin{array}{ccc} U^{-1}R \times M & \xrightarrow{B} & N \\ {\scriptstyle A}\big\downarrow & \nearrow_{\tilde{B}} & \\ U^{-1}M & & \end{array}$$

Moreover, it is unique in making this diagram commute by definition as any other mapping equal to B would be equal5 to $\tilde{B}$. ☐

Exercise 4:
Suppose R is a commutative ring and U is a multiplicative subset of R such that $1 \in U$ and $0 \notin U$.

- Let U^{-1} be the functor from **R-Mod** to U^{-1}**R-Mod** described in Ash exercises 8.5.3, and 8.5.4. In particular, for any R-module M, we have $U^{-1}(M) = U^{-1}M$.
- Let T be the functor from **R-Mod** to U^{-1}**R-Mod** that acts on objects via $T(M) := U^{-1}R \otimes_R M$ and on morphisms via the universal property of tensor products.

Find a natural transformation from U^{-1} to T and prove that it is natural.

Solution. Given an R-module M define a morphism

$$\varphi_M : U^{-1}(M) \to T(M),$$

via

$$\frac{m}{u} \mapsto \frac{1}{u} \otimes m.$$

This mapping exists in the category U^{-1}**R-Mod**, indeed: for all $r \in R, u, u' \in U$ and $m, m' \in M$ we have

$$\varphi(\frac{rm}{u}) = \frac{1}{u} \otimes rm = r(\frac{1}{u} \otimes m) = r\varphi(\frac{m}{u}),$$

$$\varphi(\frac{m}{u} + \frac{m'}{u'}) = \varphi(\frac{u'm + um'}{uu'}) = \frac{1}{uu'} \otimes u'm + um' = \frac{u'}{uu'} \otimes m + \frac{u}{uu'} \otimes m' = \varphi(\frac{m}{u}) + \varphi(\frac{m'}{u'}).$$

We will show that the family of all these morphisms for all $M \in \text{Obj}(\textbf{R-Mod})$ form a natural transformation $U^{-1} \Rightarrow T$.

Suppose $f : M \to N$ is an $R-$linear morphism between two R-modules M and N. We will show the commutativity of the following diagram:

$$\begin{array}{ccc} U^{-1}(M) & \xrightarrow{U^{-1}(f)} & U^{-1}(N) \\ {\scriptstyle \varphi_M}\big\downarrow & & \big\downarrow{\scriptstyle \varphi_N} \\ T(M) & \xrightarrow[T(f)]{} & T(N) \end{array}$$

Let $\frac{m}{u} \in U^{-1}(M)$, then by definition

$$U^{-1}(f)(\frac{m}{u}) = \frac{f(m)}{u},$$

and

$$\varphi_N(\frac{f(m)}{u}) = \frac{1}{u} \otimes f(m).$$

On the other hand we have

$$\varphi_M(\frac{m}{u}) = \frac{1}{u} \otimes m,$$

and by definition of T(f)

$$T(f)(\frac{1}{u} \otimes m) = \frac{1}{u} \otimes f(m).$$

Thus this family of morphisms provides a natural transformation we are looking for.

□

Exercise 5 (Ash 10.5.1-3):
We are going to prove the *projective basis lemma*, which states that an R-module P is projective if and only if there are elements $x_i \in P$ ($i \in I$) and homomorphisms $f_i : P \to R$ such that for every $x \in P$, $f_i(x) = 0$ for all but finitely many i and

$$x = \sum_i f_i(x)x_i.$$

The set of x_i's is referred to as the projective basis.

1. To prove the "only if" part, let P be a direct summand of the free module F with basis $\{e_i\}$. Take f to be the inclusion map of P into F, and π the natural projection of F onto P. Show how to define the f_i and x_i so that the desired results are obtained.

2. To prove the "if" part, let F be a free module with basis $\{e_i, i \in I\}$, and define $\pi : F \to P$ by $\pi(e_i) = x_i$. Define $f : P \to F$ by $f(x) = \sum_i f_i(x)e_i$. Show that πf is the identity on P.

3. Continuing Problem 2, show that P is projective.

Solution. 1."only if"

Let P be a projective module, that is a direct summand of a free module F. Let $f : P \hookrightarrow F$ be the natural inclusion and $\pi : F \twoheadrightarrow P$ the natural projection. Since F is free, is has a basis $\{e_i\}$, so that for every $m \in F$, we can write

$$m = \sum a_i e_i,$$

where all but finitely many $a_i = 0 \in R$. Let $x_i = \pi(e_i) \in P$. Now for each i, consider the dual functional to each basis element: $e_i^* : F \to R$ such that

$$e_i^*(e_j) = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}.$$

These are functions defined on a basis of a free module F, so we can extend them linearly to be an R-linear function on all of F. Given $x \in P$ define $f_i(x) = e_i^*(f(x))$, then given $x \in P$, we can write

$$f(x) = \sum e_i^*(f(x))e_i = \sum f_i(x)e_i.$$

Note that since P is projective, the natural inclusion is a right inverse to π, so that $\pi \circ f = \mathrm{id}_P$, and hence

$$x = \pi(f(x)) = \sum f_i(x)\pi(e_i) = \sum f_i(x)x_i.$$

2-3. "if" Suppose there are elements $x_i \in P$ for $i \in I$ and linear functionals $f_i : P \to R$ such that for every $x \in P$ we can write

$$x = \sum f_i(x)x_i,$$

where all but finitely many $f_i(x)$ are zero. If F is free with basis $\{e_i\}$, we can define a linear function $\pi : F \to P$ on the basis via $\pi(e_i) = x_i$. Additionally, we can define $f : P \to F$ via $f(x) = \sum f_i(x)e_i$, which is a well defined linear function: for all $x, y \in P$ and $a \in R$,

$$f(x+y) = \sum f_i(x+y)e_i = \sum f_i(x)e_i + \sum f_i(y)e_i = f(x + f(y),$$

$$f(ax) = \sum f_i(ax)e_i = a \sum f_i(x)e_i = af(x).$$

Then

$$\pi(f(x)) = \pi\left(\sum f_i(x)e_i\right) = \sum f_i(x)\pi(e_i) = \sum f_i(x)x_i = x.$$

By the above we can see that π is both surjective, so that we have the following short exact sequence:

$$0 \to \ker(\pi) \to F \to P \to 0$$

and that f acts a section, i.e. is a right inverse to π, and hence the sequence splits, so that $F \cong \ker(\pi) \oplus P$, giving that P is projective.

□