HW 2 Due: 11 February 2026

As always, your answer will be graded on the quality of presentation as well as the correct answer. To get a good score: write your answer neatly, use complete sentences, and *justify your work*.

1. Suppose that C is a category with zero objects. Choose any $A \in \mathrm{Obj}(\mathsf{C})$. Let's write 0_A for the zero morphism in $\mathrm{Hom}_{\mathsf{C}}(A, A)$ and id_A for the identity morphism in $\mathrm{Hom}_{\mathsf{C}}(A, A)$. Prove that A is a zero object if and only if $0_A = \mathrm{id}_A$.

2. Suppose that C, D are categories with zero objects, and $F\colon \mathsf{C} \to \mathsf{D}$ is a functor. Suppose that F has the property that it sends zero morphisms to zero morphisms. Prove that if Z is a zero object in $\mathrm{Obj}(\mathsf{C})$, then $F(Z)$ is a zero object in $\mathrm{Obj}(\mathsf{D})$. (Recall that functors must send identity morphisms to identity morphisms.)

3. Suppose R is a commutative ring and U is a multiplicative subset of R such that $1 \in U$ and $0 \notin U$. Prove that $U^{-1}R \otimes_R M$ and $U^{-1}M$ are isomorphic as $U^{-1}R$-modules. (See [Ash10, Exercises 8.5.3 and 8.5.4] for the definition of $U^{-1}M$.)

4. Suppose R is a commutative ring and U is a multiplicative subset of R such that $1 \in U$ and $0 \notin U$.

   - Let U^{-1} be the functor from R-Mod to $U^{-1}R$-Mod described in [Ash10, Exercises 8.5.3 and 8.5.4]. In particular, for any R-module M, we have $U^{-1}(M) = U^{-1}M$.
   - Let T be the functor from R-Mod to $U^{-1}R$-Mod that acts on objects via $T(M) \coloneqq U^{-1}R \otimes_R M$ and on morphisms via the universal property of tensor products.

   Find a natural transformation from U^{-1} to T and prove it is natural.

5. [Ash10, Problems 1–3, Section 10.5]

References

[Ash10] Robert B. Ash, *Abstract Algebra: The Basic Graduate Year*, 2010.