Exercise 1:
Since $\mathbb{C}$ is commutative, it is

(a) an $(\mathbb{R}, \mathbb{R})$ bimodule, as well as a left $\mathbb{R}$-module using restriction of scalars and

(b) an $(\mathbb{C}, \mathbb{C})$ bimodule, and a left $\mathbb{C}$-module

By (a), we can construct the $\mathbb{R}$-module $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$, and by (b) we can construct the $\mathbb{C}$-module $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$. However, using restriction of scalars, we can consider $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ as an $\mathbb{R}$-module. Show that the $\mathbb{R}$-module $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and the $\mathbb{R}$-module $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ are not isomorphic.

Solution. In what follows, the restriction of scalars via the inclusion $\iota : \mathbb{R} \hookrightarrow \mathbb{C}$ will be used implicitly. We will show that the two $\mathbb{R}$-modules ($\mathbb{R}$-vector spaces) have different dimension.

Starting with $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$, we can apply the proposition: If $\{x_1, \ldots, x_m\}$ is a basis for R^m and $\{y_1, \ldots, y_n\}$ is a basis of R^n, then $\{x_i \otimes y_j, i = 1, \ldots, m, j = 1, \ldots, n\}$ is a basis of $R^m \otimes R^n$. Since $\mathbb{C}$ is a 2 dimensional $\mathbb{R}$-module with basis $\{1, i\}$, we have that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is a 4 dimensional $\mathbb{R}$-module, with basis $\{1 \otimes 1, i \otimes 1, 1 \otimes i, i \otimes i\}$.

Now, $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \simeq \mathbb{C}$, by the proposition 8.7.6 in Ash, thus it is a two dimensional $\mathbb{R}$-vector space. □

Exercise 2:
If m and n are relatively prime, show that $\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n = 0$

Solution. Since m, n are relatively prime, there exists $a, b \in \mathbb{Z}$ such that $am + bn = 1$. Then for any elementary tensor $x \otimes y \in \mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n$ we have

$$x \otimes y = (am + bn)(x \otimes y) = x \cdot (am + bn) \otimes y = xma \otimes y + xnb \otimes y = 0 \otimes y + x \otimes bny = 0 \otimes 0.$$

Thus since the tensor product is generated by elementary tensors, an arbitrary element of $\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n$ is equal to the zero element. □

Exercise 3:
Let M, N, M', N' be arbitrary R-modules, where R is a commutative ring. Show that the tensor product of homomorphisms induces a linear map

$$\mathrm{Hom}_R(M, M') \otimes_R \mathrm{Hom}_R(N, N') \to \mathrm{Hom}_R(M \otimes_R N, M' \otimes_R N').$$

Solution. Let

$$B : \mathrm{Hom}_R(M, M') \times \mathrm{Hom}(N, N') \to \mathrm{Hom}_R(M \otimes N, M' \otimes N')$$

be given by

$$B(f_1, f_2) = f_1 \otimes f_2$$

as in the tensor product of homomorphisms. Then B is bilinear. Indeed, we have for all $r_1, r_2 \in R$, $f_1, g_1 \in \mathrm{Hom}_R(M, M'))$, and $f_2, g_2 \in \mathrm{Hom}_R(N, N')$,

$$B(r_1(f_1 + g_1), r_2(f_2 + g_2))(x_1 \otimes x_2) = r_1(f_1 + g_1)(x_1) \otimes r_2(f_2 + g_2)(x_2)$$

$$= r_1 r_2(f_1(x_1) \otimes f_2(x_2)) + r_1 r_2(g_1(x_1) \otimes f_2(x_2)) + r_1 r_2(f_1(x_1) \otimes g_2(x_2)) + r_1 r_2(g_1(x_1) \otimes g_2(x_2))$$

$$= (r_1 r_2 B(f_1, f_2) + r_1 r_2 B(g_1, f_2) + r_1 r_2 B(f_1, g_2) + r_1 r_2 B(g_1, g_2))(x_1 \otimes x_2).$$

Thus by the universal property of tensor products there exists a linear form

$$L : \mathrm{Hom}_R(M, M') \otimes_R \mathrm{Hom}_R(N, N') \to \mathrm{Hom}_R(M \otimes_R N, M' \otimes_R N'),$$

which is induced by the tensor product of homomorphisms given via B. □