HW 3 Due: 10 November 2025

As always, your answer will be graded on the quality of presentation as well as the correct answer. To get a good score: write your answer neatly, use complete sentences, and *justify your work.*

1. Let R be a commutative ring, and $R \to A$ a commutative R-algebra. We say that A is *graded* if there exist sub-R-modules of A for every nonnegative integer (let's call them $A_0, A_1, \ldots$) with the following properties:

   - A is isomorphic (as an R-module) to $A_0 \oplus A_1 \oplus \cdots$, and
   - for all $i, j \in \mathbb{Z}_{\geq 0}$ and $a_i \in A_i, a_j \in A_j$, we have $a_i a_j \in A_{i+j}$.

   If A is a graded R-algebra and $n \in \mathbb{Z}$, define $A(n)$ to be the graded R-algebra with graded pieces:

   $$\text{for any } i \in \mathbb{Z}_{\geq 0}, \qquad A(n)_i = \begin{cases} \{0\} & \text{if } i + n < 0 \\ A_{i+n} & \text{if } i + n \geq 0. \end{cases}$$

   For an ideal I of A, we say I is *graded* if $I = (I \cap A_0) \oplus (I \cap A_1) \oplus \cdots$. (Some people say *homogeneous.*)

   If $B \simeq B_0 \oplus B_1 \oplus \cdots$ is another graded R-algebra, we say a R-algebra homomorphism $\phi\colon A \to B$ is *graded* if for all $i \in \mathbb{Z}_{\geq 0}$, we have $\phi(A_i) \subseteq B_i$.

   (a) Show that $\mathbb{Z}[x]$ is a graded $\mathbb{Z}$-algebra.

   (b) Show that the principal ideal of $\mathbb{Z}[x]$ generated by $x + 1$ is not a graded ideal.

   (c) Suppose $A \simeq A_0 \oplus A_1 \oplus \cdots$ is a graded R-algebra. Choose any $i \in \mathbb{Z}_{\geq 0}$ and $a_i \in A_i$. Prove that the principal ideal of A generated by a_i is a graded ideal of A.

   (d) Suppose $A \simeq A_0 \oplus A_1 \oplus \cdots$ is a graded R-algebra and I is a graded ideal of A. For any $i \in \mathbb{Z}_{\geq 0}$, let's write I_i for $I \cap A_i$. Prove that A/I is a graded R-algebra by considering the homomorphism of graded R-algebras whose existence is guaranteed by the definition of coproduct

   $$\pi\colon A \to (A_0/I_0) \oplus (A_1/I_1) \oplus \cdots.$$

   (e) Let $A = \mathbb{C}[x, y]$, so that A is a graded $\mathbb{C}$-algebra with the grading of "total degree". Let $f = y^2 - x^3$ and I the ideal it generates. Finally let $\phi\colon A(-3) \to A$ be the function defined by $\phi(a) = fa$; prove that is a homomorphism of graded R-algebras. Then consider the following short exact sequence of graded $\mathbb{C}$-algebras

   $$0 \to A(-3) \to A \to A/I \to 0.$$

   For every $i \in \mathbb{Z}_{\geq 0}$, compute the dimension (as a $\mathbb{C}$-vector space) of the ith graded piece of A/I.

References

[Ash10] Robert B. Ash, *Abstract Algebra: The Basic Graduate Year*, 2010.