HW 2 Due: 22 October 2025

As always, your answer will be graded on the quality of presentation as well as the correct answer. To get a good score: write your answer neatly, use complete sentences, and *justify your work*.

1. Problems 2–5 from [Ash10, Section 4.3]. Then read and think about Problem 6 from [Ash10, Section 4.3] for five minutes.
2. Problem 1 from [Ash10, Section 4.4].
3. Problems 1–3 from [Ash10, Section 4.7]
4. Let $\alpha = \sqrt{-5}$ and $M = \mathbb{Z}[\alpha]$. Choose any $(a, b) \in \mathbb{Z}^2$. Define a function:

$$\begin{aligned} T_{(a,b)} \colon M &\to M \\ \beta &\mapsto (a + b\alpha)\beta. \end{aligned}$$

   (a) Prove that M is a free $\mathbb{Z}$-module by finding a $\mathbb{Z}$-basis for M and proving it is a $\mathbb{Z}$-basis.
   (b) Prove that $T_{(a,b)}$ is a $\mathbb{Z}$-module homomorphism.
   (c) What is the matrix for $T_{(a,b)}$ in terms of the basis you found in part (a)?
   (d) For which $a, b \in \mathbb{Z}$ is $T_{(a,b)}$ invertible?
   (e) Pretend I said $\alpha = \sqrt{5}$. Find at least six pairs $(a, b) \in \mathbb{Z}^2$ for which $T_{(a,b)}$ is invertible.

5. Let

$$M = \{ f \in \mathbb{R}[x] \mid \deg f < 5 \}.$$

   Define a function:

$$\begin{aligned} \partial \colon M &\to M \\ f &\mapsto f' \end{aligned}$$

   (a) Convince yourself the M is a free $\mathbb{R}$-module and write down a basis for M.
   (b) Convince yourself that ∂ is a linear transformation and write down a matrix for ∂ in terms of the basis you chose in part (a).

References

[Ash10] Robert B. Ash, *Abstract Algebra: The Basic Graduate Year*, 2010.