HW8

Math 443/5

8 June 2026

This HW is not for turning in, and serves as *preparation for the final*.

1. Let $f \in \mathbb{Q}[x]$ be an irreducible cubic, say with complex roots α, β, γ.

   (a) Prove that f is separable. (Maximum proof length allowed: once sentence.)

   Let $K = \mathbb{Q}(\alpha, \beta, \gamma)$, so we know that $K/\mathbb{Q}$ is Galois by part (a). Let

   - $G = \mathrm{Gal}\,(K/\mathbb{Q})$,
   - $\Delta = (\alpha - \beta)(\alpha - \gamma)(\beta - \gamma)$, and
   - $D = \Delta^2$.

   (b) Prove that $D \in \mathbb{Q}$.

   Let $\pi\colon G \to S_3$ be the injective group homomorphism introduced in [Sil22, Proposition 9.45].

   (c) Prove: for all $\sigma \in G$,
   $$\pi(\sigma) \text{ is even} \qquad \text{if and only if} \qquad \sigma(\Delta) = \Delta.$$

   (d) Prove:
   $$[K : \mathbb{Q}] = 3 \qquad \text{if and only if} \qquad \Delta \in \mathbb{Q}.$$

   (e) Prove: $\Delta = 2\alpha^3 + 3\alpha^2\beta - 3\alpha\beta^2 - 2\beta^3$.

   (f) (This one is just for fun.) Show that if $f(x) = x^3 + cx + d$, then
   $$D = -4c^3 - 27d^2.$$
   (Hint: write c and d in terms of α, β, then compute $-4c^3 - 27d^2$ and compare to part (e).)

   (g) Prove: if $f(x) = x^3 + cx + d$ then
   $$[K : \mathbb{Q}] = 3 \qquad \text{if and only if} \qquad -4c^3 - 27d^2 \text{ is a square of a rational number.}$$
   (Hint: use part (f).)

References

[Sil22] Joseph H. Silverman, *Abstract Algebra: An Integrated Approach*, Pure and Applied Undergraduate Texts, vol. 55, American Mathematical Society, Providence, RI, ©2022. MR 4423367