SOME SOLUTIONS TO [Lev24]

Contents

1. Chapter One

Exercise 1.2.6.1. Translate into English:

(a) $\forall x\,(E(x) \to E(x+2))$,
(b) $\forall x \exists y(\sin x = y)$,
(c) $\forall y \exists x(\sin x = y)$, and
(d) $\forall x \exists y\left(\left(x^3 = y^3\right) \to (x = y)\right)$.

Solution. (a) This question would be better if it started with "$(\forall x \in \mathbb{R})$". Anyway, let's translate the statement as "No matter which real number x you choose : if $E(x)$ happens to be true, then so is $E(x+2)$."

(b) This question would be better if it started with "$(\forall x \in \mathbb{R})\,(\exists y \in \mathbb{R})$". Anyway, let's translate the statement as "No matter which real number x you choose, there's a real number y out there with the property with $\sin x = y$."

(c) This question would be better if it started with "$(\forall y \in \mathbb{R})\,(\exists x \in \mathbb{R})$". Anyway, let's translate the statement as "No matter which real number y you choose, there's a real number x out there with the property with $\sin x = y$."

(d) This question would be better if it started with "$(\forall x \in \mathbb{R})\,(\exists y \in \mathbb{R})$". Anyway, let's translate the statement as "No matter which real number x you choose, there's a real number y out there with the property that if x^3 happens to equal y^3, then $x = y$."

□

Exercise 1.2.6.6. Consider the statement "For all natural numbers n: if n is prime, then n is solitary." You do not need to know what *solitary* means for this problem, just that it is a property that some natural numbers have and others do not.

(a) Write the converse and the contrapositive of the statement, saying which is which. Note: the original statement claims that an implication is true for all natural numbers n, and it is that implication of which we are taking the converse and contrapositive.
(b) Write the negation of the original statement. What would you need to show to prove that the statement is false?
(c) Even though you don't know whether 10 is solitary (in fact, nobody knows this), is the statement, "If 10 is prime, then 10 is solitary" true or false? Explain.
(d) It turns out that 8 is solitary. Does this tell you anything about the truth or falsity of the original statement, its converse, or its contrapositive? Explain.
(e) Assuming that the original statement is true, what can you say about the relationship between the set p of prime numbers and the set S of solitary numbers. Explain.

Solution. (a)
- Converse: "For all natural numbers n, if n is solitary, then n is prime."
- Contrapostive: "For all natural numbers n, if n is not solitary, then n is not prime."

(b) The negation is "There exists a natural number n that is both prime and nonsolitary." To prove that this negation is true, I would have to go hunting amongst the prime numbers until I found a nonsolitary one.
(c) This implication is true, since its hypothesis is false.
(d)
- The fact that 8 is solitary tells me nothing about the truth value of the original statement, since the original statement is asserting nothing about 8.
- The fact that 8 is solitary tells me nothing about the truth value of the contrapositive of the original statement, since it is logically equivalent to the original statement.
- The fact that 8 is solitary tells me the converse of the original statement is false, since it is an example of a natural number for which the hypothesis (of the converse) is true but the conclusion (of the converse) is false.

(e) If the original statement is true, then that means $P \subseteq S$, since the definition of subset is "for all $n \in \mathbb{N}$, if $n \in P$, then $n \in S$"—and this is precisely the original statement.

□

Exercise 1.3.8.6. Simplify the following statements (so that negation only appears right before variables):
(a) $\neg(P \to \neg Q)$,
(b) $(\neg P \wedge \neg Q) \to \neg(\neg Q \wedge R)$, and
(c) $\neg((P \to \neg Q) \wedge \neg(R \wedge \neg R))$.

Solution. (a) $\neg(P \to \neg Q) \equiv P \wedge \neg\neg Q \equiv P \wedge Q$,
(b) $(\neg P \wedge \neg Q) \to \neg(\neg Q \wedge R) \equiv (\neg P \wedge \neg Q) \to (\neg\neg Q \vee \neg R) \equiv (\neg P \wedge \neg Q) \to (Q \vee \neg R) \equiv (\neg P \wedge \neg Q) \to \neg R$, and
(c) $\neg((P \to \neg Q) \wedge \neg(R \wedge \neg R)) \equiv (\neg(P \to \neg Q) \vee \neg\neg(R \wedge \neg R)) \equiv ((P \wedge \neg\neg Q) \vee (R \wedge \neg R)) \equiv ((P \wedge Q) \vee (R \wedge \neg R))$.

□

Exercise 1.4.8.14. Prove that $\log 7$ is irrational.

Proof. Suppose for a contradiction that $\log 7$ is rational, so that there are integers a, b with $b > 0$ and $\log 7 = a/b$. Exponentiating, we see that

$$7^b = \left(10^{\log 7}\right)^b = \left(10^{a/b}\right)^b = 10^a.$$

There are three cases:

- If $a = 0$, we have the contradiction
$$7^b > 1 \text{ (since } b > 1) \qquad \text{and} \qquad 7^b = 10^0 = 1.$$
- If $a < 0$, we have the contradiction
$$7^b > 1 \text{ (since } b > 1) \qquad \text{and} \qquad 7^b = 10^a < 1 \text{ (since } a < 0).$$
- Finally, if $a > 0$, we have the contradiction
$$7 \mid 7^b \text{ (since } b > 1) \qquad \text{and} \qquad 7 \nmid 7^b = 10^a \text{ (by unique factorization)}.$$

□

Exercise 1.4.8.19. Suppose you are at a party with 19 of your closest friends (so including you, there are 20 people there). Explain why there must be at least two people at the party who are friends with the same number of people at the party. Assume friendship is always reciprocated.

Proof. In principle, any person at the party could be friend with a number of people between 0 and 19 (so there are 20 options).

However, if one person is friends with everybody (all 19 other people), then *nobody* friendless (ie, nobody has 0 friends); that is, everybody has between 1 and 19 friends. On the other hand, if nobody is friends with everybody, then everybody has between 0 and 18 friends.

In either case, there are only 19 options for the number of friends for any given party attendee. But since there are 20 party attendees, we know by the pigeon hole principle that two people will have the same number of friends. □

Exercise 1.5.8.10. Let X, Y be sets and $f: X \to Y$ be a function. Prove that if A, B are subsets of Y, then $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

Proof. Since this is a set equality proof, there are two parts.

- Suppose that $x \in f^{-1}(A \cap B)$. This means that $f(x) \in A \cap B$. Since $f(x) \in A \cap B$, we know that
  - $f(x) \in A$ so that $x \in f^{-1}(A)$ and
  - $f(x) \in B$ so that $y \in f^{-1}(B)$;

  that is, we see that $x \in f^{-1}(A) \cap f^{-1}(B)$.
- Next, suppose that $x \in f^{-1}(A) \cap f^{-1}(B)$. By definition of intersection:
  - we know $x \in f^{-1}(A)$, so that $f(x) \in A$, and
  - we know $x \in f^{-1}(B)$, so that $f(x) \in B$.

  Since $f(x) \in A$ and $f(x) \in B$, we know $f(x) \in A \cap B$; that is, we know $x \in f^{-1}(A \cap B)$.

□

2. Chapter Two

Exercise 2.6.7.1. Which of the following relations on the integers are reflexive?

(a) $x \sim y$ if and only if $x + y$ is odd,
(b) $x \sim y$ if and only if $x + y$ is positive,
(c) $x \sim y$ if and only if $xy \geq 0$, and
(d) $x \sim y$ if and only if xy is positive.

Solution. (a) This relation is not reflexive, since $1 \not\sim 1$.
(b) This relation is not reflexive, since $-1 \not\sim -1$.
(c) This relation is reflexive; indeed, for any integer x we know that $x^2 \geq 0$, so that $x \sim x$.
(d) This relation is not reflexive, since $0 \not\sim 0$.

□

Exercise 2.6.7.4. Define relations $R_1, R_2, \ldots, R_6$ on $S = \{1, 2, 3, 4\}$ by

(a) $R_1 = \{(2,2),(2,3),(2,4),(3,2),(3,3),(3,4)\}$,
(b) $R_2 = \{(1,1),(1,2),(2,1),(2,2),(3,3),(4,4)\}$,
(c) $R_3 = \{(2,4),(4,2)\}$,
(d) $R_4 = \{(1,2),(2,3),(3,4)\}$,
(e) $R_5 = \{(1,1),(2,2),(3,3),(4,4)\}$, and
(f) $R_6 = \{(1,3),(1,4),(2,3),(2,4),(3,1),(3,4)\}$.

Solution. (a)
- Reflexive? No.
- Irreflexive? No.
- Symmetric? No.
- Antisymmetric? Yes.
- Transitive? Yes.

(b)
- Reflexive? Yes.
- Irreflexive? No.
- Symmetric? Yes.
- Antisymmetric? No.
- Transitive? Yes.

(c)
- Reflexive? No.
- Irreflexive? Yes.
- Symmetric? Yes.
- Antisymmetric? No.
- Transitive? No.

(d)
- Reflexive? No.
- Irreflexive? Yes.
- Symmetric? No.
- Antisymmetric? Yes.
- Transitive? No.

(e)
- Reflexive? Yes.
- Irreflexive? No.
- Symmetric? Yes.
- Antisymmetric? Yes.
- Transitive? Yes.

(f)
- Reflexive? No.
- Irreflexive? Yes.
- Symmetric? No.
- Antisymmetric? No.
- Transitive? No.

□

Exercise 2.6.7.6. For the following relations on the integers, determine whether the relation is any of reflexive, irreflexive, symmetric, antisymmetric, and transitive:

(a) $x \sim y$ if and only if $x + y = 0$,
(b) $x \sim y$ if and only if $x - y \in \mathbb{Z}$,
(c) $x \sim y$ if and only if $x = 2y$, and
(d) $x \sim y$ if and only if $xy > 1$.

Solution. (a)
- Reflexive? No.
- Irreflexive? No.
- Symmetric? Yes.
- Antisymmetric? No.
- Transitive? No.

(b)
- Reflexive? Yes.
- Irreflexive? No.
- Symmetric? Yes.
- Antisymmetric? No.
- Transitive? Yes.

(c)
- Reflexive? No.
- Irreflexive? Yes.
- Symmetric? No.
- Antisymmetric? No.
- Transitive? No.

(d)
- Reflexive? No.
- Irreflexive? Yes.
- Symmetric? Yes.
- Antisymmetric? No.
- Transitive? No.

□

3. Chapter Three

Exercise 3.1.7.3. Consider the lattice paths from $(3, 4)$ to $(8, 10)$.

- How long is each such path?
- How many steps are in the x-direction?
- How many different paths are there?

Solution.
- 11 steps.
- 5 steps.
- $\binom{11}{5}$.

□

Exercise 3.1.7.7. Suppose you are ordering a calzone from *D.P. Dough*. You want 5 distinct toppings, chosen from their list of 10 vegetarian toppings.

a. How many choices do you have for your calzone?

b. How many choices do you have for your calzone if you refuse to have green pepper as one of your toppings?
c. How many choices do you have for your calzone if you *insist* on having green pepper as one of your toppings?
d. How do parts a.–c. relate to each other?

Solution. a. $\binom{10}{5}$.
b. $\binom{9}{5}$.
c. $\binom{9}{4}$.
d. Since every choice of 5 toppings either has green peppers or do not have green peppers, parts a.–c. illustrate the "Pascal's Triangle Rule": $\binom{9}{4} + \binom{9}{5} = \binom{10}{5}$.

□

Exercise 3.2.6.4. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$.
a. How many subsets of S are there?
b. How many subsets of S have $\{2, 3, 5\}$ as a subset?
c. How many subsets of S contain at least one odd number?
d. How many subsets of S contain exactly one even number?

Solution. a. 2^8.
b. 2^5.
c. $2^8 - 2^4$.
d. $\binom{4}{1}2^4$.

□

Exercise 3.2.6.8. How many length-11 bit strings have 4 or more 1s?

Solution. There are two ways of thinking of this problem, giving the two expressions

$$2^{11} - \binom{11}{0} - \binom{11}{1} - \binom{11}{2} - \binom{11}{3} = \binom{11}{4} + \binom{11}{5} + \binom{11}{6} + \binom{11}{7} + \binom{11}{8} + \binom{11}{9} + \binom{11}{10} + \binom{11}{11}.$$

□

Exercise 3.3.6.1. Suppose you have sets A and B with $|A| = 9$ and $|B| = 19$.
(a) What is the largest possible value for $|A \cap B|$?
(b) What is the smallest possible value for $|A \cap B|$?
(c) What are the possible values for $|A \cup B|$?

Solution. (a) 9, when $A \subseteq B$.
(b) 0, when $A \cap B = \varnothing$.
(c) Any integer between 19 and 28, inclusive.

□

Exercise 3.3.6.7. How many positive integers less than 1400 are multiples of 4, 7, or 9? Use the Principle of Inclusion/Exclusion.

Solution. Let
- $S = \{n \in \mathbb{Z} \mid 1 \leq n \leq 1399\}$,
- $A = \{n \in S \mid n \text{ is a multiple of } 4\}$,
- $B = \{n \in S \mid n \text{ is a multiple of } 7\}$, and
- $C = \{n \in S \mid n \text{ is a multiple of } 9\}$,

and note that
- $|A| = 349$,
- $|B| = 199$,
- $|C| = 155$,
- $|A \cap B| = 49$,
- $|A \cap C| = 38$,
- $|B \cap C| = 22$, and

- $|A \cap B \cap C| = 5$,

so the Principle of Inclusion/Exclusion tells us

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ &= 349 + 199 + 155 - 49 - 38 - 22 + 5 \\ &= 599. \end{aligned}$$

□

Exercise 3.4.6.7. How many anagrams are there of "goggles"?

Solution. 7!/3!. You can count the number of permutations of 7 symbols using the symbols "g_3, g_2, g_3", then correct by the overcounting factor 3!. □

Exercise 3.4.6.10. Choose sets A and B with $|A| = 13$ and $\{B\} = 22$.

(a) How many functions from A to B are there?
(b) How many injective functions from A to B are there?

Solution. (a) 22^{13}.
(b) $P(22, 13)$.

□

Exercise 3.5.5.4. How many integer solutions are there to the equation $x + y + z = 12$ for which

a. x, y, z are all positive?
b. x, y, z are all nonnegative?
c. x, y, z are all at least -3?

Solution. a. $\binom{9+2}{2} = \binom{11}{2}$.
b. $\binom{12+2}{2} = \binom{14}{2}$.
c. This number is the same as the number of nonnegative integer solutions to $X + Y + Z = 21$, which is $\binom{21+2}{2} = \binom{23}{2}$.

□

Exercise 3.5.5.7. How many integer solutions to $x_1 + x_2 + x_3 + x_4 = 35$ are there for which $x_1 \geq 4$, $x_2 \geq 4$, $x_3 \geq 1$, and $x_4 \geq 4$?

Solution. The answer is the same as the number of nonnegative integer solutions to

$$X_1 + X_2 + X_3 + X_4 = 22,$$

which is $\binom{25}{3}$. □

Exercise 3.6.6.6. Suppose that n, k are positive integers, with $1 \leq k \leq n$. Consider the identity

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

(a) Is it true?
(b) Give an algebraic proof.
(c) Give a combinatorial proof.

Solution. (a) Yup.
(b) Note that

$$k\binom{n}{k} = k\left(\frac{n!}{k!(n-k)!}\right) = \frac{n!}{(k-1)!(n-k)!} = n\left(\frac{(n-1)!}{(k-1)!((n-1)-(k-1))!}\right) = n\binom{n-1}{k-1}.$$

(c) The number of ways to
  - first choose k oranges from a set of n oranges
  - and then paint one of them red

  is $k\binom{n}{k}$.
  But this number is equal to the number of ways to
  - first choose one of n oranges to paint red

- and then choose $k-1$ of the remaining $n-1$ oranges;

this number is $n\binom{n-1}{k-1}$.

□

Exercise 3.6.6.7. Suppose that n, k are integers with $2 \leq k \leq n$. Give a combinatorial proof of the identity

$$\binom{n}{2}\binom{n-2}{k-2} = \binom{n}{k}\binom{k}{2}.$$

Solution. The strategy is very similar to that of Exercise 3.6.6.6. The number of ways to

- first choose k oranges from a set of n oranges
- and then paint two of them red

is $\binom{n}{k}\binom{k}{2}$.

But this number is equal to the number of ways to

- first choose two of n oranges to paint red
- and then choose $k-2$ of the remaining $n-2$ oranges;

this number is $\binom{n}{2}\binom{n-2}{k-2}$. □

4. Chapter Four

5. Chapter Five

Exercise 5.1.5.1. Let $A = \{2,4,5,6,7\}$ and $B = \{2,4,5,6,8\}$. Find each of the following sets (your answers should include the curly braces):

a. $A \cup B$,
b. $A \cap B$,
c. $A \setminus B$,
d. $B \setminus A$.

Solution. a. $A \cup B = \{2,4,5,6,7,8\}$,
b. $A \cap B = \{2,4,5,6\}$,
c. $A \setminus B = \{7\}$,
d. $B \setminus A = \{8\}$.

□

Exercise 5.1.5.2. Find the least element of the following sets, if there is one:

a. $\{n \in \mathbb{N} \mid n^2 - 3 \geq 4\}$,
b. $\{n \in \mathbb{N} \mid n^2 - 7 \in \mathbb{N}\}$,
c. $\{n^2 + 4 \mid n \in \mathbb{N}\}$,
d. $\{n \in \mathbb{N} \mid \text{there exists } k \in \mathbb{N} \text{ such that } n = k^2 + 4\}$.

Solution. a. 3,
b. 3,
c. 4,
d. 4.

□

Exercise 5.1.5.3. Find the following cardinalities.

a. $|A|$ where $A = \{4,5,6,7,\ldots,33\}$.
b. $|B|$ where $B = \{x \in \mathbb{Z} \mid -7 \leq x \leq 91\}$.
c. $|C \cap D|$ where $C = \{x \in \mathbb{N} \mid x \leq 27\}$ and $D = \{x \in \mathbb{N} \mid x \text{ is prime}\}$.

Solution. a. $|A| = 33 - 3 = 30$.
b. $|B| = 91 + 8 = 99$.
c. $|C \cap D| = |\{2,3,5,7,11,13,17,19,23\}| = 9$.

□

Exercise 5.2.4.1. Consider the function $f\colon \{1,2,3,4,5\} \to \{1,2,3,4,5,\}$ given by

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 4 & 2 & 5 \end{pmatrix}.$$

a. Find $f(4)$.
b. Find n in the domain of f such that $f(n) = 4$.
c. Find an element n in the domain of f such that $f(n) = n$.
d. Find an element in the codomain of f that is not in the range of f.

Solution. a. $f(4) = 2$.
b. $f(3) = 4$, so 3 is a good choice.
c. $f(1) = 1$ and $f(5) = 5$ so any element of $\{1,5\}$ is a good choice.
d. The only good choice is 3.

□

Exercise 5.2.4.2. The following functions all have $\{1,2,3,4,5\}$ as both their domain and codomain. For each, determine whether it is (only) injective, (only) surjective, bijective, or neither injective nor surjective.

(a) $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 1 \end{pmatrix}$.

(b) $g(x) = \begin{cases} x & \text{if } x < 3 \\ x - 2 & \text{if } x \geq 3. \end{cases}$

(c) $h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 5 & 4 \end{pmatrix}$.

(d) $i(x) = \begin{cases} 3 - x & \text{if } x < 3 \\ x & \text{if } x \geq 3. \end{cases}$

Proof. (a) The function f is neither injective nor surjective.
(b) The function g is neither injective nor surjective.
(c) The function h is bijective.
(d) The function i is bijective.

□

Exercise 5.2.4.3. The following functions all have $\{1,2,3,4,5\}$ as their domain and $\{1,2,3\}$ as their codomain. For each, determine whether it is (only) injective, (only) surjective, bijective, or neither injective nor surjective.

(a) $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 3 & 2 & 1 & 1 \end{pmatrix}$.

(b) $g(x) = \begin{cases} x & \text{if } x < 4 \\ 6 - x & \text{if } x \geq 4. \end{cases}$

(c) $h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 2 & 3 & 1 \end{pmatrix}$.

Solution. (a) The function f is surjective but not injective.
(b) The function g is surjective but not injective.
(c) The function h is surjective but not injective.

□

References

[Lev24] Oscar Levin, *Discrete Mathematics—An Open Introduction*, fourth ed., 2024.