HW 8 Due: 13 March 2025

As always, your answer will be graded on the quality of presentation as well as the correct answer. To get a good score: write your answer neatly, use complete sentences, and *justify your work.*

1. Prove that for all $n \in \mathbb{N}$,
$$1 + 3 + \cdots + (2n+1) = (n+1)^2.$$

Proof. We will use induction. To see that the base case is true, we note that $1 = (0+1)^2$.

For the induction step, assume that $k \in \mathbb{N}$ and $1 + 3 + \cdots + (2k+1) = (k+1)^2$. Note that

$$\begin{aligned}
1 + 3 + \cdots + (2k+1) + (2(k+1)+1) &= \Big(1 + 3 + \cdots + (2k+1)\Big) + (2k+3) && \text{(algebra)} \\
&= (k+1)^2 + (2k+3) && \text{(induction hypothesis)} \\
&= k^2 + 4k + 4 && \text{(algebra)} \\
&= (k+2)^2 && \text{(more algebra).}
\end{aligned}$$

□

2. (a) Prove: for all $n \in \mathbb{Z}_{\geq 3}$, $n^2 > 2n+1$.
   (b) Prove: for all $n \in \mathbb{Z}_{\geq 5}$, $2^n > n^2$.

Proof. (a) We will use induction. To see that the base case is true, we note that $3^2 = 9 > 7 = 2(3)+1$. For the induction step, assume that $k \in \mathbb{Z}_{\geq 3}$ and $k^2 > 2k+1$. Note that

$$\begin{aligned}
(k+1)^2 &= k^2 + 2k + 1 && \text{(algebra)} \\
&> (2k+1) + 2k + 1 && \text{(induction hypothesis)} \\
&= 2(k+1) + 2k && \text{(algebra)} \\
&> 2(k+1) + 1 && \text{(since } k \geq 3\text{).}
\end{aligned}$$

(b) We will use induction. To see that the base case is true, we note that $2^5 = 32 > 25 = 5^2$. For the induction step, assume that $k \in \mathbb{Z}_{\geq 5}$ and $2^k > k^2$. Note that

$$\begin{aligned}
2^{k+1} &= 2\left(2^k\right) && \text{(algebra)} \\
&> 2\left(k^2\right) && \text{(induction hypothesis)} \\
&= (k+1)^2 + k^2 - 2k - 1 && \text{(algebra)} \\
&> (k+1)^2 + (2k+1) - 2k - 1 && \text{(by part (a))} \\
&> (k+1)^2 && \text{(more algebra).}
\end{aligned}$$

□

3. Let α be a nonzero real number with the property that $\alpha + \frac{1}{\alpha}$ is an integer. Prove
$$\text{for all } n \in \mathbb{N},\ \alpha^n + \frac{1}{\alpha^n} \text{ is an integer.}$$

Proof. We will use induction. To see that the base case is true, we note that since $\alpha \neq 0$,
$$\alpha^0 + \frac{1}{\alpha^0} = 2 \in \mathbb{Z}.$$

Moreover, we know that $\alpha^1 + \frac{1}{\alpha^1} \in \mathbb{Z}$ by hypothesis.

For the (strong) induction step, assume that $k \in \mathbb{Z}_{\geq 1}$ and for all $j \in \{0, \ldots, k\}$,

$$\alpha^j + \frac{1}{\alpha^j} \in \mathbb{Z}.$$

Note that

$$\begin{aligned}
&\alpha^{k+1} + \frac{1}{\alpha^{k+1}} \\
&\quad = \left(\alpha^k + \frac{1}{\alpha^k}\right)\left(\alpha + \frac{1}{\alpha}\right) - \left(\alpha^{k-1} + \frac{1}{\alpha^{k-1}}\right) && \text{(algebra)} \\
&\quad \in \mathbb{Z} && \text{(use the strong induction hypothesis with } j = k, k-1, 1\text{)}.
\end{aligned}$$

□

4. Define a sequence (a_n) by setting $a_0 = 1$, and for all $n \in \mathbb{N}$:

$$a_{n+1} = a_n - \frac{1}{2}a_n^2.$$

Then define a new sequence (b_n) by setting, for all $n \in \mathbb{N}$,

$$b_n = \frac{2}{a_n}.$$

(a) Prove that for all $n \in \mathbb{N}$, $0 < a_n \leq 1$. (Thanks to this part, the sequence (b_n) is well-defined, phew.)

(b) Prove that for all $n \in \mathbb{N}$, $b_n \geq n + 2$.

Proof. (a) We will use induction. To see that the base case is true, we note that $0 < 1 \leq 1$.

For the induction step, assume that $k \in \mathbb{N}$ and $0 < a_k \leq 1$. Note that

$$\begin{aligned}
a_{k+1} &= a_k - \frac{1}{2}a_k^2 && \text{(definition of } (a_n)\text{)} \\
&\leq 1 - 0 && \text{(squares are positive)}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
a_{k+1} &= a_k - \frac{1}{2}a_k^2 && \text{(definition of } (a_n)\text{)} \\
&= a_k\left(1 - \frac{1}{2}a_k\right) && \text{(algebra)} \\
&> a_k\left(1 - \frac{1}{2}(1)\right) && \text{(since } a_k > 0 \text{ and } a_k \leq 1 \text{ by the induction hypothesis)} \\
&= \frac{a_k}{2} && \text{(algebra)} \\
&> 0 && \text{(since } a_k > 0 \text{ by the induction hypothesis)}.
\end{aligned}$$

(b) We will use induction. To see that the base case is true, we note that $b_0 = \frac{2}{1} = 2 \geq 0 + 2$. For the induction step, assume that $k \in \mathbb{N}$ and $b_k \geq k + 2$. Note that

$$
\begin{aligned}
b_{k+1} &= \frac{2}{a_{k+1}} && \text{(definition of } (b_n)\text{)} \\
&= \frac{2}{a_k - \frac{1}{2}a_k^2} && \text{(definition of } (a_n)\text{)} \\
&= \frac{4}{a_k(2 - a_k)} && \text{(algebra)} \\
&= \frac{4 - 2a_k + 2a_k - a_k^2 + a_k^2}{a_k(2 - a_k)} && \text{(algebra)} \\
&= \frac{4 - 2a_k}{a_k(2 - a_k)} + \frac{2a_k - a_k^2}{a_k(2 - a_k)} + \frac{a_k^2}{a_k(2 - a_k)} && \text{(algebra)} \\
&= \frac{2}{a_k} + 1 + \frac{a_k}{2 - a_k} && \text{(algebra)} \\
&= \frac{2}{a_k} + 1 + \frac{1}{\frac{2}{a_k} - 1} && \text{(algebra)} \\
&= b_k + 1 + \frac{1}{b_k - 1} && \text{(definition of } (b_n)\text{)} \\
&\geq (k + 2) + 1 + \frac{1}{b_k - 1} && \text{(induction hypothesis)} \\
&= k + 3 + \frac{1}{b_k - 1} && \text{(algebra)} \\
&> k + 3 + 0 && \text{(the induction hypothesis tells us } b_k - 1 > 0\text{)}.
\end{aligned}
$$

□