# HW 7 Due: 6 March 2025

As always, your answer will be graded on the quality of presentation as well as the correct answer. To get a good score: write your answer neatly, use complete sentences, and *justify your work*.

1. Suppose $(a_n)_{n\in\mathbb{N}}$ is a sequence with $a_0 = 5$, and for any $n \in \mathbb{N}$,

$$a_{n+1} = a_n + n(n-2).$$

Find a closed formula for $(a_n)_{n\in\mathbb{N}}$.

*Solution.* We compute the first few terms

$$(a_n)_{n\in\mathbb{N}} = 5, 5, 4, 4, 7, 15, 30, \ldots$$

and notice that

- the sequence of first differences is $0, -1, 0, 3, 8, 15, \ldots$,
- the sequence of second differences is $-1, 1, 3, 5, 7 \ldots$,
- and the sequence of third differences is $2, 2, 2, 2 \ldots$,

so $(a_n)_{n\in\mathbb{N}}$ is a $\Delta^3$ sequence. By [Lev24, Theorem 4.3.5], there are $a, b, c, d \in \mathbb{R}$ such that for all $n \in \mathbb{N}$,

$$a_n = an^3 + bn^2 + cn + d.$$

In particular,

$$\begin{array}{ccccccc} 5 & = & a_0 & = & a0^3 + b0^2 + c0 + d & = & 0a + 0b + 0c + 1d \\ 5 & = & a_1 & = & a1^3 + b1^2 + c1 + d & = & 1a + 1b + 1c + 1d \\ 4 & = & a_2 & = & a2^3 + b2^2 + c2 + d & = & 8a + 4b + 2c + 1d \\ 4 & = & a_3 & = & a3^3 + b3^2 + c3 + d & = & 27a + 9b + 3c + 1d, \end{array}$$

so we solve the augmented matrix

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 5 \\ 1 & 1 & 1 & 1 & 5 \\ 8 & 4 & 2 & 1 & 4 \\ 27 & 9 & 3 & 1 & 4 \end{array}\right]$$

to see that $a = \frac{1}{3}, b = -\frac{3}{2}, c = \frac{7}{6}, d = 5$. That is, for all $n \in \mathbb{N}$,

$$a_n = \frac{1}{3}n^3 - \frac{3}{2}n^2 + \frac{7}{6}n + 5.$$

□

2. Suppose $(b_n)_{n\in\mathbb{N}}$ is a sequence with $b_4 = 17$, and for any $n \in \mathbb{N}$,

$$b_{n+1} = b_n + (n+1)^2.$$

Find a closed formula for $(b_n)_{n\in\mathbb{N}}$.

*Solution.* Notice that

- $17 = b_4 = b_3 + 4^2$, so that $b_3 = 1$,
- $1 = b_3 = b_2 + 3^2$, so that $b_2 = -8$,
- $-8 = b_2 = b_1 + 2^2$, so that $b_1 = -12$, and

- $-12 = b_1 = b_2 + 1^2$, so that $b_0 = -13$,

so the first few terms of the sequence are

$$(b_n)_{n \in \mathbb{N}} = -13, -12, -8, 1, 17, 42, 78, \ldots.$$

Notice that

- the sequence of first differences is $1, 4, 9, 16, 25, 36, \ldots$,
- the sequence of second differences is $3, 5, 7, 9, 11 \ldots$,
- and the sequence of third differences is $2, 2, 2, 2 \ldots$,

so $(a_n)_{n \in \mathbb{N}}$ is a $\Delta^3$ sequence. By [Lev24, Theorem 4.3.5], there are $a, b, c, d \in \mathbb{R}$ such that for all $n \in \mathbb{N}$,

$$b_n = an^3 + bn^2 + cn + d.$$

In particular,

$$\begin{aligned}
-13 &= b_0 = a0^3 + b0^2 + c0 + d = 0a + 0b + 0c + 1d \\
-12 &= b_1 = a1^3 + b1^2 + c1 + d = 1a + 1b + 1c + 1d \\
-8 &= b_2 = a2^3 + b2^2 + c2 + d = 8a + 4b + 2c + 1d \\
1 &= b_3 = a3^3 + b3^2 + c3 + d = 27a + 9b + 3c + 1d,
\end{aligned}$$

so we solve the augmented matrix

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 1 & -13 \\ 1 & 1 & 1 & 1 & -12 \\ 8 & 4 & 2 & 1 & -8 \\ 27 & 9 & 3 & 1 & 1 \end{array}\right]$$

to see that $a = \frac{1}{3}, b = \frac{1}{2}, c = \frac{1}{6}, d = -13$. That is, for all $n \in \mathbb{N}$,

$$a_n = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n - 13.$$

□

3. Suppose $(c_n)_{n \in \mathbb{N}}$ is a sequence with $c_0 = 5$, $c_1 = 28$, and for any $n \in \mathbb{N}$,

$$c_{n+2} = 11c_{n+1} - 30c_n.$$

Find a closed formula for $(c_n)_{n \in \mathbb{N}}$.

*Solution.* This sequence has characteristic polynomial $x^2 - 11x + 30 = (x-6)(x-5)$, so we know by [Lev24, Section 4.4] that there are $a, b \in \mathbb{R}$ such that for all $n \in \mathbb{N}$,

$$c_n = a(6)^n + b(5)^n.$$

In particular,

$$\begin{aligned}
5 &= c_0 = a6^0 + b5^0 = 1a + 1b \\
28 &= c_1 = a6^1 + b5^1 = 6a + 5b,
\end{aligned}$$

so we solve the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 6 & 5 & 28 \end{array}\right]$$

to see that $a = 3, b = 2$. That is, for all $n \in \mathbb{N}$,

$$c_n = 3(6)^n + 2(5)^n.$$

□

4. Consider the sequence $(d_n)_{n\in\mathbb{N}}$ defined by $d_n = 3\left(1+\sqrt{5}\right)^n + 3\left(1-\sqrt{5}\right)^n$. Find integers $A, B$ such that for all $n \in \mathbb{N}$,

$$d_{n+2} = Ad_{n+1} + Bd_n.$$

*Solution.* By [Lev24, Section 4.4], since $(d_n)_{n\in\mathbb{N}}$ is an exponential sequence, it has a characteristic polynomial

$$\left(x - \left(1+\sqrt{5}\right)\right)\left(x - \left(1-\sqrt{5}\right)\right) = x^2 - (1-\sqrt{5})x - (1+\sqrt{5})x + (1+\sqrt{5})(1-\sqrt{5}) = x^2 - 2x - 4.$$

Thus, by [Lev24, Section 4.4] we take $A = 2, B = 4$. □

# References


[Lev24] Oscar Levin, *Discrete Mathematics—An Open Introduction*, fourth ed., 2024.