HW 6 — Due: 27 February 2025

As always, your answer will be graded on the quality of presentation as well as the correct answer. To get a good score: write your answer neatly, use complete sentences, and *justify your work*.

1. For any $n \in \mathbb{N}$, let p_n be the number of lattice paths from (n, n) to $(n^2, n+2)$.
   (a) What is p_4?
   (b) Write down a closed formula for p_n.

Solution. (a) $p_4 = \binom{(16-4)+(6-4)}{6-4} = \binom{14}{2}$

(b) $p_n = \binom{(n^2-n)+(n+2-n)}{n+2-n} = \binom{n^2-n+2}{2}$. □

2. (a) Let $a_0 = 1$, and for any $n \in \mathbb{Z}_{\geq 1}$, let $a_n = a_{n-1}^2 + 1$. What is a_6?
   (b) Prove: for any $d \in \mathbb{Z}$ and $n \in \mathbb{N}$, if $d \mid a_n$ and $d \mid a_{n+1}$, then $d = \pm 1$.

Solution. (a)

$$
\begin{aligned}
a_0 & & &= 1 \\
a_1 & &= 1^2 + 1 &= 2 \\
a_2 & &= 2^2 + 1 &= 5 \\
a_3 & &= 5^2 + 1 &= 26 \\
a_4 & &= 26^2 + 1 &= 677 \\
a_5 & &= 677^2 + 1 &= 458330 \\
a_6 & &= 458330^2 + 1 &= 210066388901
\end{aligned}
$$

(b) Choose any $d \in \mathbb{Z}$ and $n \in \mathbb{N}$ and suppose that $d \mid a_n$ and $d \mid a_{n+1}$. By the definition of divide, there exist $j, k \in \mathbb{Z}$ such that

$$
\begin{aligned}
a_n &= dj \\
a_{n+1} &= dk.
\end{aligned}
$$

Then by the definition of $(a_n)_{n \in \mathbb{N}}$, we see

$$dk = a_{n+1} = \left(a_n^2\right) + 1 = (dj)^2 + 1 = d^2 j^2 + 1,$$

so

$$1 = dk - d^2 j^2 = d\left(k - dj^2\right).$$

That is, we see that $d \mid 1$, so that $d = \pm 1$. □

3. Suppose that $(a_n)_{n \in \mathbb{N}}$ is an arithmetic sequence. If $a_4 = 32$ and $a_7 = 53$, what are a_5 and a_6?

Solution. $a_5 = 39$ and $a_6 = 46$. □

4. Let $(a_n)_{n\in\mathbb{N}}$ be the arithmetic sequence $a_n = 5n + 2$. For which primes p and nonnegative integers n does p divide both a_n and a_{n+1}?

   Proof. We will prove that there are no such primes p and nonnegative integers n.

   For our first step, we will prove

$$(\forall \text{ primes } p)(\forall n \in \mathbb{N})\Big(((p \mid a_n) \wedge (p \mid a_{n+1})) \to (p = 5)\Big).$$

   To do so, choose any prime p and nonnegative integer n, and suppose that $p \mid a_n$ and $p \mid a_{n+1}$. By the definition of divide, there exist $j, k \in \mathbb{Z}$ such that

$$a_n = pj$$
$$a_{n+1} = pk.$$

   Then by the definition of $(a_n)_{n\in\mathbb{N}}$, we see

$$pk = a_{n+1} = a_n + 5 = pj + 5,$$

   so

$$5 = pk - pj = p(k - j).$$

   That is, we see that $p \mid 5$, so that $p = 5$ since p is prime.

   We will conclude by proving

$$(\forall n \in \mathbb{N})(5 \nmid a_n).$$

   To do so, suppose for a contradiction that there exists $n \in \mathbb{N}$ such that $5 \mid a_n$. Then by definition of divide, there exists $\ell \in \mathbb{Z}$ such that $a_n = 5\ell$. Using the definition of $(a_n)_{n\in\mathbb{N}}$, this means that

$$5\ell = a_n = 5n + 2,$$

   so that $2 = 5\ell - 5n = 5(\ell - n)$, so that $5 \mid 2$ by the definition of divide. But of course we know that $5 \nmid 2$, so we have achieved our contradiction. □

5. Suppose that $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are geometric sequences. Suppose that for all $n \in \mathbb{Z}_{\geq 0}$, we know $b_n \neq 0$. Prove that the sequence $(c_n)_{n\in\mathbb{N}}$ defined by $c_n = a_n/b_n$ is also geometric.

   Proof. By the definition from [Lev24, Section 4.2.3], we know there are common ratios $r \in \mathbb{R}$ and $s \in \mathbb{R}$ for $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$, respectively. Since we know that for all $n \in \mathbb{N}$, the real number b_n is nonzero, we know that $s \neq 0$. In particular, we know that $r/s \in \mathbb{R}$.

   To show that $(c_n)_{n\in\mathbb{N}}$ is geometric, we will show it has common ratio r/s. To this end, choose any $n \in \mathbb{N}$. Then

$$c_{n+1} = \frac{a_{n+1}}{b_{n+1}} = \frac{ra_n}{sb_n} = \left(\frac{r}{s}\right)\left(\frac{a_n}{b_n}\right) = \left(\frac{r}{s}\right)c_n.$$

□

References

[Lev24] Oscar Levin, *Discrete Mathematics—An Open Introduction*, fourth ed., 2024.