HW 6 Due: 27 February 2025

As always, your answer will be graded on the quality of presentation as well as the correct answer. To get a good score: write your answer neatly, use complete sentences, and *justify your work*.

1. For any $n \in \mathbb{N}$, let p_n be the number of lattice paths from (n, n) to $(n^2, n+2)$.
   (a) What is p_4?
   (b) Write down a closed formula for p_n.
2. (a) Let $a_0 = 1$, and for any $n \in \mathbb{Z}_{\geq 1}$, let $a_n = a_{n-1}^2 + 1$. What is a_6?
   (b) Prove: for any $d \in \mathbb{Z}$ and $n \in \mathbb{N}$, if $d \mid a_n$ and $d \mid a_{n+1}$, then $d = \pm 1$.
3. Suppose that $(a_n)_{n \in \mathbb{N}}$ is an arithmetic sequence. If $a_4 = 32$ and $a_7 = 53$, what are a_5 and a_6?
4. Let $(a_n)_{n \in \mathbb{N}}$ be the arithmetic sequence $a_n = 5n + 2$. For which primes p and nonnegative integers n does p divide both a_n and a_{n+1}?
5. Suppose that $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ are geometric sequences. Suppose that for all $n \in \mathbb{Z}_{\geq 0}$, we know $b_n \neq 0$. Prove that the sequence $(c_n)_{n \in \mathbb{N}}$ defined by $c_n = a_n / b_n$ is also geometric.