Math 356 Winter 2025 Midterm

Name:____

- Put your name in the "______________________________________" above.
- Answer at least five problems, your **best five** will count.
- Proofs are graded for both clarity and rigor.
- Good luck!

1. (a) Write a truth table for $(p \vee q) \to (p \wedge q)$.

P	Q	$P \vee Q$	$P \wedge Q$	$P \vee Q \to P \wedge Q$
T	T	T	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	F	T

(b) Write an expression that is logically equivalent to the statement above but does not use an implication arrow ($\to$) or a bi-implication arrow ($\leftrightarrow$).

$$(P \wedge Q) \vee \left((\neg P) \wedge (\neg Q)\right)$$

(c) Let $P(x, y)$ be the predicate "$x^2 + 1 \geq y^2$. What are the truth values of the following? (No proof necessary.)

i. $(\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z})(P(x, y))$ T $(y = 0)$

ii. $(\exists x \in \mathbb{Z})(\forall y \in \mathbb{Z})(P(x, y))$ F

iii. $(\forall y \in \mathbb{Z})(\exists x \in \mathbb{Z})(P(x, y))$ T

iv. $(\exists y \in \mathbb{Z})(\forall x \in \mathbb{Z})(P(x, y))$ T $(y = 0)$

2. This problem is about counting ways to walk from the origin of the xy-plan using only

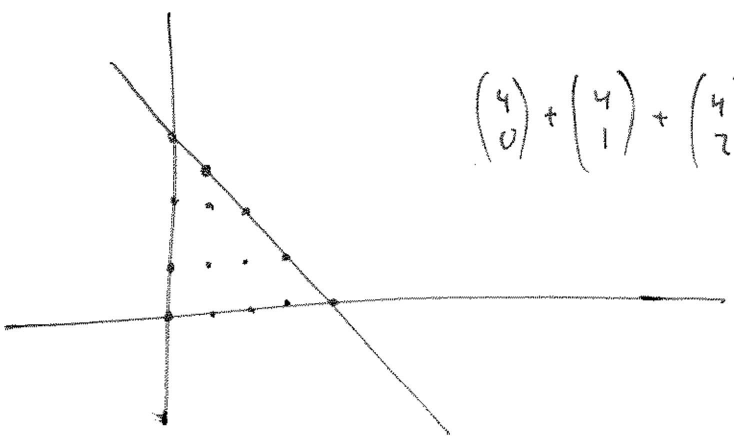
- "right step", moving to the right a distance of one, and
- "up step", moving up a distance of one.

Answers to the following questions may contain binomial coefficients (that is, expressions of the form $\binom{n}{m}$ for nonnegative integers n, m with $n \geq m$.)

(a) How many ways are there to walk from the point $(1,1)$ to the point $(12,19)$, passing through the point $(6,7)$?

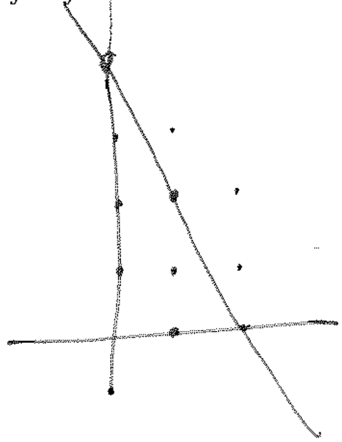
$$\binom{11}{5}\binom{18}{6}$$

(b) How many ways are there to walk from from the point $(0,0)$ to a point on the line $y = -x + 4$?



$$\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4}$$

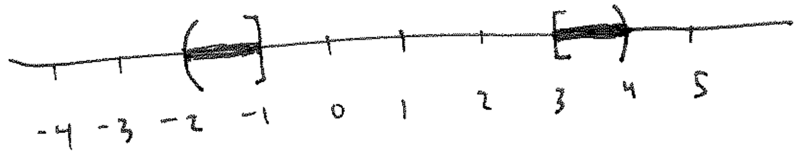
(c) How many ways are there to walk from from the point $(0,0)$ to a point on the line $y = -2x + 4$?



$$\binom{4}{0} + \binom{3}{1} + \binom{2}{2}$$

3. For every $n \in \mathbb{Z}_{\geq 0}$, let $C_n = \{x \in \mathbb{R} \mid n - 1 \leq |x - 1| < n\}$, and let $\mathcal{C} = \{C_n \mid n \in \mathbb{Z}_{\geq 0}\}$.

(a) Draw a picture of C_3.



(b) Prove that $\mathcal{C}$ is not a partition of $\mathbb{R}$.

Note that $C_0 = \{ x \in \mathbb{R} \mid -1 \leq |x-1| < 0 \}$

$= \emptyset$,

since no real numbers have negative absolute value.

Thus, $\mathcal{C}$ is not a partition. $\square$

(c) You can remove one set from $\mathcal{C}$ to make a partition of $\mathbb{R}$. Which one? (No proof necessary.)

C_0

4. How many integer solutions to $w + x + y + z = 20$ are there

(a) with w, x, y, z nonnegative?

$$\binom{23}{3}$$

(b) with w, x, y, z nonnegative and $w \leq 3$?

$$\binom{23}{3} - \binom{19}{3}$$

– or –

$$\binom{22}{2} + \binom{21}{2} + \binom{20}{2} + \binom{19}{2}$$

(c) with w, x, y, z all nonnegative and even?

$$\binom{13}{3}$$

5. Let $A = \{1,2,3,4\}$ and $B = \{3,4,5,6\}$.

(a) Compute $|A \cup B|$.

$|A \cup B| = |\{1,2,3,4,5,6\}| = 6$

(b) Compute $|A \cap B|$.

$|A \cap B| = |\{3,4\}| = 2$

(c) Write down all elements of $(A \cap B) \times (A \cap B)$.

$(3,3),\ (3,4),\ (4,3),\ (4,4)$

(d) For any set S, define $\mathcal{P}(S)$ to be the set of all subsets of S. Write down all elements of $\mathcal{P}(A) \cap \mathcal{P}(B)$.

$\emptyset,\ \{3\},\ \{4\},\ \{3,4\}$

6. Recall that for any integers a, b we say that a *divides* b when there is some integer c such that $ac = b$. Prove that for all integers n,

if 3 does not divide $(n-1)^2 + 2$, then 3 does not divide n.

We will prove this statement using the contrapositive, so suppose that 3 divides n. Then there is some integer c such that $n = 3c$, so that

$$
\begin{aligned}
(n-1)^2 + 2 &= (3c-1)^2 + 2 \\
&= 9c^2 - 6c + 1 + 2 \\
&= 9c^2 - 6c + 3 \\
&= 3(3c^2 - 2c + 1),
\end{aligned}
$$

and we see $(n-1)^2 + 2$ is divisible by 3 by definition. $\square$

7. We say that two positive integers a, b are *multiplicatively independent* whenever

$$\forall m, n \in \mathbb{Z}_{\geq 0} \left((a^m = b^n) \to ((m = 0) \wedge (n = 0)) \right).$$

(a) Negate (and simplify) the above quantified statement.

$$\exists\, m, n \in \mathbb{Z} \left((a^m = b^n) \wedge \left((m \neq 0) \vee (n \neq 0) \right) \right)$$

(b) Prove that for all integers a, the pair $a, 1$ are not multiplicatively independent.

note that $a^0 = 1 = 1^2$, so we may take $m = 0$ and $n = 2$.

(c) Find a pair of distinct integers, both greater than one, which are not multiplicatively independent.

Consider $a = 100$
$b = 1000$.

Then $100^3 = (10^2)^3 = 10^6 = (10^3)^2 = 1000^2$,

so $100, 1000$ are not multiplicatively independent.

Extra credit (if you have extra time)

Suppose that a, b are integers greater than one. Prove that for all $a, b \in \mathbb{Z}_{\geq 2}$,

$$a, b \text{ are not multiplicatively independent} \quad \text{if and only if} \quad \frac{\ln a}{\ln b} \in \mathbb{Q}$$

$\boxed{\Rightarrow}$ Suppose a, b are not multiplicatively independent. Then there are $m, n \in \mathbb{Z}_{\geq 0}$ such that $a^m = b^n$ and either m or n are nonzero.

If m nonzero, then the fact that $a > 1$ implies $b^n = a^m > 1$, so n nonzero as well. Similarly, if n nonzero, then m is as well. That is, both m, n are nonzero and

$$a^m = b^n$$
$$\ln(a^m) = \ln(b^n)$$
$$m \ln a = n \ln b, \quad \text{so}$$

$$\frac{\ln a}{\ln b} = \frac{n}{m} \in \mathbb{Q}.$$

$\Leftarrow$ Suppose $\frac{\ln a}{\ln b} \in \mathbb{Q}$, so there exist $m, n \in \mathbb{Z}$ such that $m \neq 0$ and $\frac{\ln a}{\ln b} = \frac{n}{m}$.

Since $a, b \geq 2$, we know $\ln a, \ln b > 0$, so we may assume $m, n > 0$. Then

$$\frac{\ln a}{\ln b} = \frac{n}{m}$$
$$m \ln a = n \ln b$$
$$\ln a^m = \ln b^n$$
$$a^m = b^n,$$

so a, b not multiplicatively independent. $\square$