Math 356 Winter 2025 Midterm

Name:____

- Put your name in the "____" above.
- Answer at least five problems, your **best five** will count.
- Proofs are graded for both clarity and rigor.
- Good luck!

1. (a) Write a truth table for $(p \vee q) \rightarrow (p \wedge q)$.

   (b) Write an expression that is logically equivalent to the statement above but does not use an implication arrow ($\rightarrow$) or a bi-implication arrow ($\leftrightarrow$).

   (c) Let $P(x, y)$ be the predicate "$x^2 + 1 \geq y^2$. What are the truth values of the following? (No proof necessary.)

      i. $(\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z})(P(x, y))$

      ii. $(\exists x \in \mathbb{Z})(\forall y \in \mathbb{Z})(P(x, y))$

      iii. $(\forall y \in \mathbb{Z})(\exists x \in \mathbb{Z})(P(x, y))$

      iv. $(\exists y \in \mathbb{Z})(\forall x \in \mathbb{Z})(P(x, y))$

2. This problem is about counting ways to walk from the origin of the xy-plan using only

   - "right step", moving to the right a distance of one, and
   - "up step", moving up a distance of one.

   Answers to the following questions may contain binomial coefficients (that is, expressions of the form $\binom{n}{m}$ for nonnegative integers n, m with $n \geq m$.)

   (a) How many ways are there to walk from the point $(1,1)$ to the point $(12,19)$, passing through the point $(6,7)$?

   (b) How many ways are there to walk from from the point $(0,0)$ to a point on the line $y = -x + 4$?

   (c) How many ways are there to walk from from the point $(0,0)$ to a point on the line $y = -2x + 4$?

3. For every $n \in \mathbb{Z}_{\geq 0}$, let $C_n = \{x \in \mathbb{R} \mid n - 1 \leq |x - 1| < n\}$, and let $\mathcal{C} = \{C_n \mid n \in \mathbb{Z}_{\geq 0}\}$.

   (a) Draw a picture of C_3.

   (b) Prove that $\mathcal{C}$ is not a partition of $\mathbb{R}$.

   (c) You can remove one set from $\mathcal{C}$ to make a partition of $\mathbb{R}$. Which one? (No proof necessary.)

4. How many integer solutions to $w + x + y + z = 20$ are there

   (a) with w, x, y, z nonnegative?

   (b) with w, x, y, z nonnegative and $w \leq 3$?

   (c) with w, x, y, z all nonnegative and even?

5. Let $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$.

(a) Compute $|A \cup B|$.

(b) Compute $|A \cap B|$.

(c) Write down all elements of $(A \cap B) \times (A \cap B)$.

(d) For any set S, define $\mathcal{P}(S)$ to be the set of all subsets of S. Write down all elements of $\mathcal{P}(A) \cap \mathcal{P}(B)$.

6. Recall that for any integers a, b we say that a *divides* b when there is some integer c such that $ac = b$. Prove that for all integers n,

$$\text{if 3 does not divide } (n-1)^2 + 2, \text{ then 3 does not divide } n.$$

7. We say that two positive integers a, b are *multiplicatively independent* whenever

$$\forall m, n \in \mathbb{Z}_{\geq 0} \left((a^m = b^n) \rightarrow ((m = 0) \wedge (n = 0)) \right).$$

(a) Negate (and simplify) the above quantified statement.

(b) Prove that for all integers a, the pair $a, 1$ are not multiplicatively independent.

(c) Find a pair of distinct integers, both greater than one, which are not multiplicatively independent.

Extra credit (if you have extra time)

Suppose that a, b are integers greater than one. Prove that for all $a, b \in \mathbb{Z}_{\geq 2}$,

$$a, b \text{ are not multiplicatively independent} \qquad \text{if and only if} \qquad \frac{\ln a}{\ln b} \in \mathbb{Q}$$