Math 356 Winter 2023 Midterm

Name:____________________________________

- Put your name in the "____________________________________" above.
- Answer at least five problems, your **best five** will count.
- Proofs are graded for clarity, rigor, neatness, and style.
- Good luck!

1. (a) Write a truth table for $p \wedge (p \to q)$.

p	q	$p \to q$	$p \wedge (p \to q)$
T	T	T	T
T	F	F	F
F	T	T	F
F	F	T	F

(b) Write an expression that is logically equivalent to the statement above but does not use an implication arrow ($\to$) or a bi-implication arrow ($\leftrightarrow$).

$p \wedge q$

(c) Recall that for $a, b \in \mathbb{Z}$, we write $a \mid b$ if there exists some $c \in \mathbb{Z}$ such that $ac = b$. Let $P(a,b)$ be the predicate "$a \mid (b^2+3)$". What are the truth values of the following? (No proof necessary.)

i. $(\forall a \in \mathbb{Z})(\exists b \in \mathbb{Z})(P(a,b))$ F $(a=0)$

ii. $(\exists a \in \mathbb{Z})(\forall b \in \mathbb{Z})(P(a,b))$ T $(a=1)$

iii. $(\forall b \in \mathbb{Z})(\exists a \in \mathbb{Z})(P(a,b))$ T $(a=1)$

iv. $(\exists b \in \mathbb{Z})(\forall a \in \mathbb{Z})(P(a,b))$ F

2. This problem is about counting ways to walk from the origin of the xy-plan using only

- "right step", moving to the right a distance of one, and
- "up step", moving up a distance of one.

Answers to the following questions may contain binomial coefficients (that is, expressions of the form $\binom{n}{m}$ for nonnegative integers n, m with $n \geq m$.)

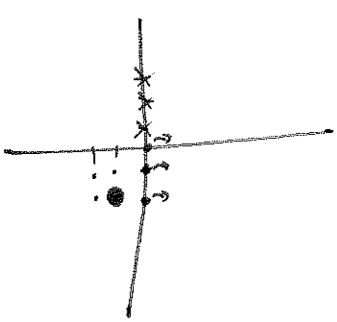
(a) How many ways are there to walk from the point $(-1,-2)$ to the point $(12,19)$?

$$\binom{34}{13}$$

(b) How many ways are there to walk from from the point $(-1,-2)$ to the point (12,19) without passing through the point (4,5)?

$$\binom{34}{13} - \binom{12}{5}\binom{22}{8}$$

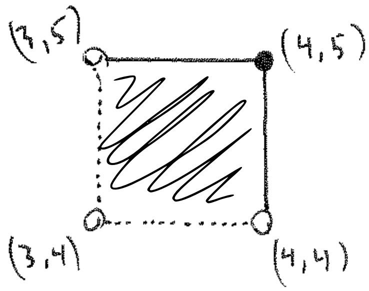
(c) How many ways are there to walk from from the point $(-1,-2)$ to the point (12,19) without stepping on the positive y-axis? (In other words, without stepping on any point in $\{(0,k) \mid k \in \mathbb{Z}_{>0}\}$?)



$$\binom{1}{1}\binom{32}{11} + \binom{2}{1}\binom{31}{11} + \binom{3}{1}\binom{30}{11}$$

3. For every $(a,b) \in \mathbb{Z}^2$, let $C_{(a,b)} = \{(x,y) \in \mathbb{R}^2 \mid a < x \leq a+1 \text{ and } b < y \leq b+1\}$, and let $C = \{C_{(a,b)} \mid (a,b) \in \mathbb{Z}^2\}$.

(a) Draw a picture of $C_{(3,4)}$.



(b) If you had to prove that C is a partition of $\mathbb{R}^2$, you would have to prove three things. Prove *one* of the three things (any one you like).

- Choose any $(x,y) \in \mathbb{R}^2$. Let $a, b \in \mathbb{Z}$ be the largest integers such that $a < x$ and $b < y$, respectively. But this means $a+1 \geq x$ and $b+1 \geq y$, so $(x,y) \in C_{(a,b)}$. Thus $\bigcup_{(a,b) \in \mathbb{Z}^2} C_{(a,b)} = \mathbb{R}^2$.

- Choose ~~and~~ any $(a,b) \in \mathbb{Z}^2$. Note that $(a+1, b+1) \in C_{(a,b)}$, so that $C_{(a,b)} \neq \emptyset$.

- Choose any $(a,b), (c,d) \in \mathbb{Z}^2$ with $(a,b) \neq (c,d)$.

  Case 1: $a \neq c$, so let's say $a < c$. This means that $a \leq c-1$, so if $(x,y) \in C_{(a,b)}$, then $x \leq a+1 \leq (c-1)+1 = c$, so $(x,y) \notin C_{(c,d)}$.

  Case 2: $b \neq d$ is similar.

4. Prove that for all $n \in \mathbb{Z}_{>0}$, there exist $a, b \in \mathbb{Z}$ such that $5^n = a^2 + b^2$. (Hint: do two base cases, and for the induction step, choose some $n \in \mathbb{Z}_{>2}$ and use strong induction to use the $n-2$ case.)

We will use strong induction.

Base Case: Note that $5^1 = 5 = 1+4$ and $5^2 = 25 = 9+16$,

so the base case is true.

Induction Step: $\S$ $n \in \mathbb{Z}_{>2}$ and for all $j \in \{1, \dots, n-1\}$, we can write 5^j as the sum of two squares. By our strong induction hypothesis, there exist $a, b \in \mathbb{Z}$ such that $5^{n-2} = a^2+b^2$. Then

$$\begin{aligned} 5^n &= 5^2 5^{n-2} \\ &= 5^2(a^2+b^2) \\ &= (5a)^2 + (5b)^2, \end{aligned}$$

as desired. $\square$

5. Let A be the set of even numbers between 0 and 18 (inclusive) and let B be the set of multiples of 3 between 0 and 18 (inclusive).

$A = \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18\}$

$B = \{0, 3, 6, 9, 12, 15, 18\}$

(a) Compute $|A \cup B|$.

13

(b) Compute $|A \cap B|$.

4

(c) Write down all elements of $(A \cap B) \times (A \cap B)$.

(0,0) (0,6) (0,12) (0,18)

(6,0) (6,6) (6,12) (6,18)

(12,0) (12,6) (12,12) (12,18)

(18,0) (18,6) (18,12) (18,18)

(d) Write down all elements of $\mathcal{P}(B \setminus A)$. $B \setminus A = \{3, 9, 15\}$, so

$\mathcal{P}(B \setminus A) = \{\emptyset, \{3\}, \{9\}, \{15\}, \{3,9\}, \{3,15\}, \{9,15\}, \{3,9,15\}\}$

(e) Suppose the universe consists of integers between 0 and 18 (inclusive). Write down all elements of $A \cap B^c$.

2, 4, 8, 10, 14, 16

6. Prove that for all $x \in \mathbb{R}$,

$$\text{if } x^2 - 10x + 1 < 0, \text{ then } x > 0.$$

We will use contraposition. Suppose $x \in \mathbb{R}$ and $x \leq 0$.
Multiplying by -10 yields $-10x \geq 0$, so

$$\begin{aligned} x^2 - 10x + 1 &= x^2 + (-10x) + 1 \\ &\geq x^2 + 0 + 1 && \text{(by above)} \\ &= x^2 + 1 \\ &> 0 && (\text{since } x^2 \geq 0), \end{aligned}$$

as desired. $\square$

7. If $S \subseteq \mathbb{R}$ and $x \in \mathbb{R}$, we say that x is a *limit point* of S whenever

$$(\forall \epsilon \in \mathbb{R}_{>0})\,(\exists s \in S)\,(|x - s| < \epsilon)\,.$$

(a) Negate (and simplify) the above quantified statement.

$$(\exists \varepsilon \in \mathbb{R}_{>0})(\forall s \in S)(|x-s| \geq \varepsilon)$$

(b) Let $T = \{\frac{1}{n} \mid n \in \mathbb{Z}_{>0}\}$. Is 0 a limit point of T? (No proof necessary.)

yes

(c) Prove that for all $S \subseteq \mathbb{R}$ and for all $x \in \mathbb{R}$: if $x \in S$, then x is a limit point of S.

Assume $x \in S$ and choose any $\varepsilon \in \mathbb{R}_{>0}$. Since $x \in S$, we may take $s = x$. Then we see that $|x-s| = |x-x| = 0 < \varepsilon$, as desired. $\square$

Extra credit (if you have extra time)

Prove your answer to 7.(b).

Choose any $\varepsilon \in \mathbb{R}_{>0}$. Since $(\frac{1}{n})$ converges to 0 (it is the harmonic sequence), there exists some $n \in \mathbb{Z}_{>0}$ such that $\frac{1}{n} < \varepsilon$. We can take $s = \frac{1}{n}$, so that

$|0 - s| = |-\frac{1}{n}| = \frac{1}{n} < \varepsilon$, as desired. $\square$