Math 356 Winter 2023 Midterm

Name:____

- Put your name in the "____" above.
- Answer at least five problems, your **best five** will count.
- Proofs are graded for clarity, rigor, neatness, and style.
- Good luck!

1. (a) Write a truth table for $p \wedge (p \to q)$.

   (b) Write an expression that is logically equivalent to the statement above but does not use an implication arrow ($\to$) or a bi-implication arrow ($\leftrightarrow$).

   (c) Recall that for $a, b \in \mathbb{Z}$, we write $a \mid b$ if there exists some $c \in \mathbb{Z}$ such that $ac = b$. Let $P(a,b)$ be the predicate "$a \mid (b^2 + 3)$". What are the truth values of the following? (No proof necessary.)

      i. $(\forall a \in \mathbb{Z})(\exists b \in \mathbb{Z})(P(a,b))$

      ii. $(\exists a \in \mathbb{Z})(\forall b \in \mathbb{Z})(P(a,b))$

      iii. $(\forall b \in \mathbb{Z})(\exists a \in \mathbb{Z})(P(a,b))$

      iv. $(\exists b \in \mathbb{Z})(\forall a \in \mathbb{Z})(P(a,b))$

2. This problem is about counting ways to walk from the origin of the xy-plan using only

   - "right step", moving to the right a distance of one, and
   - "up step", moving up a distance of one.

   Answers to the following questions may contain binomial coefficients (that is, expressions of the form $\binom{n}{m}$ for nonnegative integers n, m with $n \geq m$.)

   (a) How many ways are there to walk from the point $(-1, -2)$ to the point $(12, 19)$?

   (b) How many ways are there to walk from from the point $(-1, -2)$ to the point (12,19) without passing through the point (4,5)?

   (c) How many ways are there to walk from from the point $(-1, -2)$ to the point (12,19) without stepping on the positive y-axis? (In other words, without stepping on any point in $\{(0, k) \mid k \in \mathbb{Z}_{>0}\}$?)

3. For every $(a,b) \in \mathbb{Z}^2$, let $C_{(a,b)} = \{(x,y) \in \mathbb{R}^2 \mid a < x \leq a+1 \text{ and } b < y \leq b+1\}$, and let $\mathcal{C} = \{C_{(a,b)} \mid (a,b) \in \mathbb{Z}^2\}$.

(a) Draw a picture of $C_{(3,4)}$.

(b) If you had to prove that $\mathcal{C}$ is a partition of $\mathbb{R}^2$, you would have to prove three things. Prove *one* of the three things (any one you like).

4. Prove that for all $n \in \mathbb{Z}_{>0}$, there exist $a, b \in \mathbb{Z}$ such that $5^n = a^2 + b^2$. (Hint: do two base cases, and for the induction step, choose some $n \in \mathbb{Z}_{>2}$ and use strong induction to use the $n - 2$ case.)

5. Let A be the set of even numbers between 0 and 18 (inclusive) and let B be the set of multiples of 3 between 0 and 18 (inclusive).

   (a) Compute $|A \cup B|$.

   (b) Compute $|A \cap B|$.

   (c) Write down all elements of $(A \cap B) \times (A \cap B)$.

   (d) Write down all elements of $\mathcal{P}(B \smallsetminus A)$.

   (e) Suppose the universe consists of integers between 0 and 18 (inclusive). Write down all elements of $A \cap B^c$.

6. Prove that for all $x \in \mathbb{R}$,

$$\text{if } x^2 - 10x + 1 < 0, \text{ then } x > 0.$$

7. If $S \subseteq \mathbb{R}$ and $x \in \mathbb{R}$, we say that x is a *limit point* of S whenever

$$(\forall \epsilon \in \mathbb{R}_{>0})\,(\exists s \in S)\,(|x - s| < \epsilon)\,.$$

(a) Negate (and simplify) the above quantified statement.

(b) Let $T = \left\{\frac{1}{n} \mid n \in \mathbb{Z}_{>0}\right\}$. Is 0 a limit point of T? (No proof necessary.)

(c) Prove that for all $S \subseteq \mathbb{R}$ and for all $x \in \mathbb{R}$: if $x \in S$, then x is a limit point of S.

Extra credit (if you have extra time)

Prove your answer to 7.(b).