Math 356 Winter 2023 Final

Name:____

- Put your name in the "____" above.
- Answer at least five problems, your **best five** will count.
- Proofs are graded for clarity, rigor, neatness, and style.
- Good luck!

1. Define the subset L of $\mathbb{Z} \times \mathbb{Z}$ by setting

$$L = \{(2a, 2b) \mid a, b \in \mathbb{Z}\}.$$

Define the relation $\sim$ on $\mathbb{Z} \times \mathbb{Z}$ by the rule

$$(a, b) \sim (c, d) \quad \text{if and only if} \quad (a - c, b - d) \in L.$$

(a) Prove that $\sim$ is an equivalence relation.

Proof.
- To see that $\sim$ is reflexive, choose any $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ and note that $(a - a, b - b) = (0, 0) = (2 \cdot 0, 2 \cdot 0)$, so $(a, b) \sim (a, b)$.
- To see that $\sim$ is symmetric, choose any $(a, b), (c, d) \in \mathbb{Z} \times \mathbb{Z}$ and assume that $(a, b) \sim (c, d)$. By the definition of $\sim$, there are $m, n \in \mathbb{Z}$ such that $(a - c, b - d) = (2m, 2n)$. Note that $(c - a, d - b) = (-(a - c), -(b - d)) = (-(2m), -(2n)) = (2(-m), 2(-n))$, so $(c, d) \sim (a, b)$.
- To see that $\sim$ is transitive, choose any $(a, b), (c, d), (e, f) \in \mathbb{Z} \times \mathbb{Z}$ and assume that $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. By the definition of $\sim$, there are $k, \ell, m, n \in \mathbb{Z}$ such that $(a - c, b - d) = (2k, 2\ell)$ and $(c - e, d - f) = (2m, 2n)$. Note that $(a - e, b - f) = ((a - c) + (c - e), (b - d) + (d - f)) = (2k + 2m, 2\ell + 2n) = (2(k + m), 2(\ell + n))$, so $(a, b) \sim (e, f)$. □

(b) Write down an element from every equivalence class of this equivalence relation.

Solution. $(0, 0), (1, 0), (0, 1), (1, 1)$ □

2. Let's make a password by rearranging all of the letters in the word PASSWORD.

(a) How many arrangements are possible?

(b) How many if the two S's cannot be next to each other?

(c) How many that don't begin with the letter P?

Solution. (a) $\frac{8!}{2!}$.

(b) $\frac{8!}{2!} - 7!$.

(c) $\frac{8!}{2!} - \frac{7!}{2!}$

□

3. Let $S = \{2, 4, 5, 6, 9, 10, 15, 30, 36, 48, 50, 60\}$, and consider the partial order on S given by divisibility.

(a) Draw the Hasse diagram.

(b) List all maximum elements.

(c) List all maximal elements.

(d) List all minimum elements.

(e) List all minimal elements.

(f) List all lower bounds of $\{2,9\}$.

(g) List all upper bounds of $\{2,9\}$.

4. If the following functions are invertible, state their inverse. If they are not invertible, prove they are not.

   (a) $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=|x|$.

   (b) $g:\mathbb{R}\smallsetminus\{-1\}\to\mathbb{R}\smallsetminus\{1\}$ defined by $g(x)=\frac{x-2}{x+1}$.

   (c) $h:\mathcal{P}(\{1,2,3,4,5\})\to\mathcal{P}(\{1,2,3,4,5\})$ defined by $h(S)=\{1,2,3,4,5\}\smallsetminus S$.

   Solution. (a) Since $f(-1)=1=f(1)$, we see that f is not injective.

   (b) The inverse of g is the function $j:\mathbb{R}\smallsetminus\{1\}\to\mathbb{R}\smallsetminus\{-1\}$ defined by $j(x)=\frac{x+2}{1-x}$.

   (c) The function h is its own inverse. □

5. Suppose that A is a set of size 7 and B is a set of size 10.

   (a) How many functions are there from A to B?

   (b) How many relations are there from A to B?

   (c) How many 1-1 functions are there from A to B?

   (d) How many onto functions are there from A to B?

   (e) How many symmetric relations are there on A?

   (f) How many reflexive relations are there on A?

   Solution. (a) 10^7.

   (b) 2^{70}.

   (c) $10*9*8*7*6*5*4$.

   (d) 0.

   (e) $2^{1+2+3+4+5+6+7}=2^{28}$.

   (f) 2^{42}. □

6. Let $A=\{a,b\}$ and let F be the set of functions from A to $\mathbb{Z}$. Prove that F is countable. (Hint: we know lots of sets are countable—you can show any of these sets are in bijection with F.)

   Proof. We know from class that $\mathbb{Z}\times\mathbb{Z}$ is countable, since $\mathbb{Z}$ is countable. We will prove that the cardinality of F is equal to the cardinality of $\mathbb{Z}\times\mathbb{Z}$. To do so, define

$$\begin{aligned}\phi: F &\to \mathbb{Z}\times\mathbb{Z}\\ f &\mapsto (f(a),f(b)).\end{aligned}$$

   - To see that ϕ is surjective, choose any $(m,n)\in\mathbb{Z}\times\mathbb{Z}$ and define a function $g:A\to\mathbb{Z}$ by $g(a)=m$ and $g(b)=n$. Then $\phi(g)=(g(a),g(b))=(m,n)$.
   - To see that ϕ is injective, choose any $g,h\in F$ and assume that $\phi(g)=\phi(h)$. Let's say that $\phi(g)=(m,n)=\phi(h)$ for $m,n\in\mathbb{Z}$ To prove that $g=h$, we note that the definition of ϕ tells us $g(a)=m=h(a)$ and $g(b)=n=h(b)$. □

7. Find the coefficient of b^n in the expansion of $\left(2a^2+b\right)^{2n+1}$. Don't simplify!

   Solution. $2^{n+1}a^{2(n+1)}\binom{2n+1}{n+1}$. □