HW 8 Not for turning in

As always, your answer will be graded on the quality of presentation as well as the correct answer. To get a good score: write your answer neatly, use complete sentences, and *justify your work*.

Computations

1. Consider the function

$$\begin{aligned}\phi\colon (\mathbb{Z}/9\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z}) &\to \mathbb{Z}/3\mathbb{Z}\\ (9\mathbb{Z}+m, 3\mathbb{Z}+n) &\mapsto 3\mathbb{Z}+n.\end{aligned}$$

   We know by Exercise (IV)(a) that ϕ is a homomorphism of groups. Let's write G for $(\mathbb{Z}/9\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z})$ and H for $\ker(\phi)$.

   (a) Enumerate all elements of H

   (b) Enumerate all elements of G/H.

   Solution. (a)

$$H = \{(9\mathbb{Z}, 3\mathbb{Z}), (9\mathbb{Z}+1, 3\mathbb{Z}), (9\mathbb{Z}+2, 3\mathbb{Z}), (9\mathbb{Z}+3, 3\mathbb{Z}), (9\mathbb{Z}+4, 3\mathbb{Z}), (9\mathbb{Z}+5, 3\mathbb{Z}), (9\mathbb{Z}+6, 3\mathbb{Z}), (9\mathbb{Z}+7, 3\mathbb{Z}), (9\mathbb{Z}+8, 3\mathbb{Z})\}$$

   (b)

$$G/H = \{H, H + (9\mathbb{Z}, 3\mathbb{Z}+1), H + (9\mathbb{Z}, 3\mathbb{Z}+2)\}$$

   □

2. Enumerate all elements of $4\mathbb{Z}/8\mathbb{Z}$.

   Solution. $4\mathbb{Z}/8\mathbb{Z} = \{8\mathbb{Z}+0, 8\mathbb{Z}+4\}$. □

Proofs

(I) Suppose G, H is a group are groups and $\phi\colon G \to H$ is a homomorphism, say with kernel K.

   (a) Prove: if G is abelian and ϕ is surjective, then H is abelian.

   (b) Find a counterexample to disprove the statement in the previous part if the hypothesis that ϕ is surjective is removed.

   (c) Prove: if H is abelian, then G/K is abelian.

   Proof. (a) Choose any $h_1, h_2 \in H$. Since ϕ is surjective, there are $g_1, g_2 \in G$ with $\phi(g_1) = h_1$ and $\phi(g_2) = h_2$. Then

$$\begin{aligned}h_1 h_2 &= \phi(g_1)\phi(g_2) && \text{(this was the job of } g_1, g_2\text{)}\\ &= \phi(g_1 g_2) && \text{(since } \phi \text{ is a homomorphism)}\\ &= \phi(g_2 g_1) && \text{(since } G \text{ is abelian)}\\ &= \phi(g_2)(g_1) && \text{(since } \phi \text{ is a homomorphism)}\\ &= h_1 h_2 && \text{(this was the job of } g_1, g_2\text{)}\end{aligned}$$

   (b) Consider the homomorphism from $\{0\}$ to D_4 that sends 0 to I.

   (c) By the Fundamental Homomorphism Theorem, we know that G/K is isomorphic to $\phi(G)$. Since $\phi(G)$ is a subgroup of H and H is abelian, so is $\phi(G)$. We have shown (HW4, Exercise (III)) that the fact that G/K is isomorphic to an abelian group implies that G/K itself is abelian.

   □

(II) Suppose that G is a group and H is a normal subgroup of G. Let K be *any* subgroup of G, and write

$$HK = \{hk \mid h \in H \text{ and } k \in K\}.$$

(a) Prove that HK is a subgroup of G.

(b) Prove that H is a normal subgroup of HK.

(c) Prove that for all $k \in K$, the coset Hk is an element of HK/H.

(d) Define

$$\begin{aligned} \sigma : K &\to HK/H \\ k &\mapsto Hk. \end{aligned}$$

Prove that σ is a surjective homomorphism.

(e) Prove that $\ker(\sigma) = H \cap K$.

(f) Apply the fundamental homomorphism theorem to conclude that $K/(H \cap K)$ is isomorphic to HK/H.

Proof. Let's write e for the identity of G.

(a) We must show three things.

- We know from class that $e \in H$ and $e \in K$, so $e = ee \in HK$, and we see $HK \neq \varnothing$.
- Now suppose that $h_1, h_2 \in H$ and $k_1, k_2 \in K$, so that $h_1k_1, h_2k_2 \in HK$ are arbitrary. Then

$$h_1k_1h_2k_2 = h_1\left(k_1h_2k_1^{-1}\right)(k_1k_2) \in HK$$

since
  - the fact that H is normal implies $k_1h_2k_1^{-1} \in H$,
  - and the fact that H, K are subgroups implies $h_1\left(k_1h_2k_1^{-1}\right) \in H$ and $(k_1k_2) \in K$.
- Finally choose any $h \in H$ and $k \in K$, so that $hk \in HK$ is arbitrary. Then

$$\begin{aligned} (hk)^{-1} &= k^{-1}h^{-1} && \text{(shoes and socks)} \\ &= \left(k^{-1}h^{-1}k\right)k^{-1} && \text{(definition of inverse)} \\ &\in HK && \text{(since } H \text{ is normal).} \end{aligned}$$

Thus, we see that HK is a subgroup of G.

(b) First, we know from class that e is in K, so for all $h \in H$, it follows that $h = he$ is in HK, so we see that H is indeed a subset of HK. Since H is a subgroup of G, we know it is

- nonempty,
- closed under the operation of G (which is the same as the operation of HK), and
- closed under inverses,

so H is a subgroup of HK. Finally for any $h \in H$ and $x \in HK$, we know that $xhx^{-1} \in H$, since H is normal in G (and x is in particular in G). That is, we see that H is normal, as desired.

(c) Choose any $k \in K$. We know from class that $e \in H$, so use the definition of the identity element to see that $k = ek$ is in HK, by the definition of HK. Thus, we see that $Hk = Hek \in HK/K$, as desired.

(d) To see that σ is a homomorphism, choose any $k_1, k_2 \in K$ and note that

$$\begin{aligned} \sigma(k_1k_2) &= Hk_1k_2 && \text{(definition of } \sigma) \\ &= (Hk_1)(Hk_2) && \text{(definition of the operation on } HK/K) \\ &= \sigma(k_1)\sigma(k_2) && \text{(definition of } \sigma), \end{aligned}$$

so σ is a homomorphism.

To see that σ is surjective, choose any $h \in H$ and $k \in K$, so $hk \in HK$ is arbitrary. Note that since $e \in H$, we know that $hk = ehk \in Hhk$. Since $h \in H$, we see that hk is *also* in Hk. Since cosets partition a group, and $hk \in Hhk \cap Hk$, we see that $Hk = Hhk$. Thus,

$$\begin{aligned} \sigma(k) &= Hk && \text{(definition of } \sigma\text{)} \\ &= Hhk && \text{(by the work above),} \end{aligned}$$

so σ is surjective.

(e) Note that

$$\begin{aligned} \ker(\sigma) &= \{k \in K \mid \sigma(k) = H\} && \text{(definition of kernel)} \\ &= \{k \in K \mid Hk = H\} && \text{(definition of } \sigma\text{)} \\ &= \{k \in K \mid k \in H\} && \text{(fact from HW)} \\ &= H \cap K && \text{(definition of intersection),} \end{aligned}$$

as desired.

(f) Since $\sigma \colon K \to HK/H$ is a surjective homomorphism with kernel $H \cap K$, the FHT implies $K/(H \cap K)$ is isomorphic to HK/K, as desired.

□

(III) Let G be a group with a subgroup H. Prove H is normal if and only if for all $g \in G$, we have $gH = Hg$.

Proof.

- First let's suppose that H is normal. Choose any $g \in G$.
  - Choose any $h \in H$, so that $gh \in Hg$ is arbitrary. The fact that H is a normal subgroup tells us that $gh = gh\left(g^{-1}g\right) = \left(ghg^{-1}\right)g \in Hg$.
  - Choose any $h \in H$, so that $hg \in Hg$ is arbitrary. The fact that H is a normal subgroup tells us that $hg = \left(gg^{-1}\right)hg = g\left(g^{-1}hg\right) \in gH$.
- Now suppose that for all $g \in G$, we have $gH = Hg$. To show that H is normal, choose any $g \in G$ and $h \in H$. Use our hypothesis to find an $h_0 \in H$ with $gh = h_0 g$ and note that $ghg^{-1} = (h_0 g)\,g^{-1} = h_0 \in H$.

□