

As always, your answer will be graded on the quality of presentation as well as the correct answer. To get a good score: write your answer neatly, use complete sentences, and *justify your work*.

Computations

1. Consider the function

$$\begin{aligned}\phi: (\mathbb{Z}/9\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z}) &\rightarrow \mathbb{Z}/3\mathbb{Z} \\ (9\mathbb{Z} + m, 3\mathbb{Z} + n) &\mapsto 3\mathbb{Z} + n.\end{aligned}$$

We know by HW7 that ϕ is a homomorphism. Let's write G for $(\mathbb{Z}/9\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z})$ and H for $\ker(\phi)$.

- (a) Enumerate all elements of H
 - (b) Enumerate all elements of G/H .
2. Enumerate all elements of $4\mathbb{Z}/8\mathbb{Z}$.

Proofs

- (I) Suppose G, H are groups and $\phi: G \rightarrow H$ is a homomorphism, say with kernel K .
 - (a) Prove: if G is abelian and ϕ is surjective, then H is abelian.
 - (b) Find a counterexample to disprove the statement in the previous part if the hypothesis that G is abelian is removed.
 - (c) Prove: if H is abelian, then G/K is abelian.
- (II) Suppose that G is a group and H is a normal subgroup of G . Let K be *any* subgroup of G , and write

$$HK = \{hk \mid h \in H \text{ and } k \in K\}.$$

- (a) Prove that HK is a subgroup of G .
- (b) Prove that H is a normal subgroup of HK .
- (c) Prove that for all $k \in K$, the coset Hk is an element of HK/H .
- (d) Define

$$\begin{aligned}\sigma: K &\rightarrow HK/H \\ k &\mapsto Hk.\end{aligned}$$

Prove that σ is a surjective homomorphism.

- (e) Prove that $\ker(\sigma) = H \cap K$.
 - (f) Apply the fundamental homomorphism theorem to conclude that $K/(H \cap K)$ is isomorphic to HK/H .
- (III) Let G be a group with a subgroup H . Prove H is normal if and only if for all $g \in G$, we have $gH = Hg$.