

As always, your answer will be graded on the quality of presentation as well as the correct answer. To get a good score: write your answer neatly, use complete sentences, and *justify your work*.

Computations

1. For any group G and element $g \in G$, define the homomorphism

$$e_{G,g}: \mathbb{Z} \rightarrow G \\ n \mapsto g^n.$$

(No need to prove that $e_{G,g}$ is a homomorphism.) Let's write $e_{G,g}(\mathbb{Z}) = \{e_{G,g}(n) \mid n \in \mathbb{Z}\}$. Enumerate elements of $e_{G,g}(\mathbb{Z})$ and find $\ker(e_{G,g})$ in the following situations:

- (a) $G = D_4, g = R$,
- (b) $G = \mathbb{Z}_{20}, g = 15$,
- (c) $G = D_4, g = F$.
- (d) $G = S_9, g = \text{id}_{\{1,2,3,4,5,6,7,8,9\}}$.

Solution. (a) $e_{G,g}(\mathbb{Z}) = \{I, R, R^2, R^3\}$, $\ker(e_{G,g}) = 4\mathbb{Z}$,

(b) $e_{G,g}(\mathbb{Z}) = \{0, 5, 10, 15\}$, $\ker(e_{G,g}) = 4\mathbb{Z}$,

(c) $e_{G,g}(\mathbb{Z}) = \{I, F\}$, $\ker(e_{G,g}) = 2\mathbb{Z}$, and

(d) $e_{G,g}(\mathbb{Z}) = \{\text{id}_{\{1,2,3,4,5,6,7,8,9\}}\}$, $\ker(e_{G,g}) = \mathbb{Z}$.

□

2. If G is a subgroup with a subgroup H , let's write G/H for the set $\{Hg \mid g \in G\}$. Recall that G/H is a partition of G .

- (a) Enumerate all elements of $4\mathbb{Z}/8\mathbb{Z}$.
- (b) Enumerate all elements of $D_5/\langle R \rangle$.

Solution. (a) $4\mathbb{Z}/8\mathbb{Z} = \{4\mathbb{Z} + 0, 4\mathbb{Z} + 1, 4\mathbb{Z} + 2, 4\mathbb{Z} + 3\}$.

(b) $D_5/\langle R \rangle = \{D_5I, D_5F\}$.

□

Proofs

- (I) Suppose that G, H are groups and $\psi: G \rightarrow H$ is a homomorphism.

- (a) For G' a subgroup of G , let's write $\psi(G') = \{h \in H \mid \text{there exists } g \in G' \text{ with } \psi(g) = h\}$. Prove that for all subgroups G' of G , $\psi(G')$ is a subgroup of H .
- (b) For H' a subgroup of H , let's write $\psi^{-1}(H') = \{g \in G \mid \psi(g) \in H'\}$. Prove that for all subgroups H' of H , $\psi^{-1}(H')$ is a subgroup of G .

Proof. Let's write e_G, e_H for the identities of G, H , respectively.

- (a) We must show three things to apply the subgroup test.
 - From class we know that $e_G \in G'$ and $\psi(e_G) = e_H$, so that $e_H \in \psi(G')$ and $\psi(G') \neq \emptyset$.
 - Suppose that $h_1, h_2 \in \psi(G')$. Then there are $g_1, g_2 \in G'$ with $\psi(g_1) = h_1$ and $\psi(g_2) = h_2$. Since ψ is a homomorphism, we know that $h_1 h_2 = \psi(g_1) \psi(g_2) = \psi(g_1 g_2)$, which is in $\psi(G')$ since $g_1 g_2 \in G'$ because G' is a subgroup.

- Finally, choose any $h \in \psi(G')$, so there is some $g \in G'$ with $\psi(g) = h$. Since G' is a subgroup, we know $g^{-1} \in G'$. Then from class we know that $h^{-1} = (\psi(g))^{-1} = \psi(g^{-1}) \in \psi(G)$.

(b) As in the previous part, we must show three things.

- From class we know that $\psi(e_G) = e_H \in H'$, so that $e_G \in \psi^{-1}(H')$ and $\psi^{-1}(H') \neq \emptyset$.
- Suppose that $g_1, g_2 \in \psi^{-1}(H')$. Since ψ is a homomorphism and H' is a subgroup (which is closed under the operation of H), we know that $\psi(g_1 g_2) = \psi(g_1) \psi(g_2) \in H'$, so that $g_1 g_2 \in \psi^{-1}(H')$.
- Finally, choose any $g \in \psi^{-1}(H')$. From class, and since H' is a subgroup (which is closed under inverses), we know that $\psi(g^{-1}) = (\psi(g))^{-1} \in H'$, so that $g^{-1} \in \psi^{-1}(H')$.

□

(II) Suppose that G is a commutative group, and let n be a positive integer. Define

$$\begin{aligned} \phi: G &\rightarrow G \\ g &\mapsto g^n. \end{aligned}$$

- (a) Prove that ϕ is a homomorphism of groups.
 (b) Prove that $\ker(\phi) = \{g \in G \mid \text{ord}(g) \text{ is a divisor of } n\}$.
 (c) Suppose that G is finite. Suppose further that $|G|$ and n have no common factors greater than 1. Prove that ϕ is an isomorphism.

Proof. Let's write e for the identity of G .

- (a) Choose any $g, h \in G$, and note that

$$\begin{aligned} \phi(gh) &= (gh)^n \\ &= g^n h^n && \text{(since } G \text{ is commutative)} \\ &= \phi(g)\phi(h), \end{aligned}$$

so we see that ϕ is a homomorphism.

(b) We must show two inclusions.

- Suppose that $g \in \ker(\phi)$, so that $g^n = \phi(g) = e$. By fact from class, we know that $\text{ord}(g)$ is a divisor of n .
 - Now suppose that $g \in G$ and $\text{ord}(g)$ is a divisor of n : let's say there's some $m \in \mathbb{Z}_{>0}$ with $n = \text{ord}(g) \cdot m$. Then $\phi(g) = g^n = (g^{\text{ord}(g)})^m = e^m = e$, so $g \in \ker(\phi)$.
- (c) Since G is finite, we need only show that ϕ is injective to deduce that it is bijective; the result will then follow from part (a). Well, from class we know we can show ϕ is injective by showing $\ker(\phi) = \{e\}$; to this end, suppose that $g \in \ker(\phi)$. Then $g^n = \phi(g) = e$, so that $\text{ord}(g)$ is a divisor of n . But by Lagrange's theorem, we know that $\text{ord}(g)$ is also a divisor of $|G|$. By hypothesis, we see that $\text{ord}(g) = 1$; that is, we see that $g = e$, as desired.

□

(III) Suppose that G_1, G_2 are groups, and define

$$\begin{aligned} \phi: G_1 \times G_2 &\rightarrow G_2 \\ (g_1, g_2) &\mapsto g_2. \end{aligned}$$

- (a) Prove that ϕ is a homomorphism of groups.
 (b) Prove that $\ker(\phi)$ is isomorphic to G_1 .

Proof. (a) Choose any $(g_1, g_2), (h_1, h_2) \in G_1 \times G_2$ and note that

$$\phi((g_1, g_2)(h_1, h_2)) = \phi(g_1 h_1, g_2 h_2) = g_2 h_2 = \phi(g_1, g_2) \phi(g_1, g_2).$$

(b) Let's write e_1, e_2 for the identities of G_1, G_2 , respectively. I claim that for any $g_1 \in G_1$, the element (g_1, e_2) is in $\ker(\phi)$; indeed, note that $\phi(g_1, e_2) = e_2$ by the definition of ϕ . Thus, we may define

$$\begin{aligned} \psi: G_1 &\rightarrow \ker(\phi) \\ g_1 &\mapsto (g_1, e_2). \end{aligned}$$

- To see that ψ is a homomorphism, choose any $g_1, h_1 \in G_1$ and note that

$$\psi(g_1 h_1) = (g_1 h_1, e_2) = (g_1, e_2)(h_1, e_2) = \psi(g_1) \psi(h_1).$$

- To see that ψ is injective, choose any $g_1 \in \ker(\psi)$. Then $(e_1, e_2) = \psi(g_1) = (g_1, e_2)$, so we see $g_1 = e_1$. By fact from class, this means that ψ is injective.
- Finally, to see that ψ is surjective, choose any $(g_1, g_2) \in \ker(\phi)$. This means that $e_2 = \phi(g_1, g_2) = g_2$, so that

$$\psi(g_1) = (g_1, e_2) = (g_1, g_2). \quad \square$$

(IV) Suppose G is a group and H is a normal subgroup of G .

- (a) Prove: if G is abelian, then G/H is abelian.
- (b) Prove: if $g \in G$ has finite order, then $\text{ord}(Hg)$ is a divisor of $\text{ord}(g)$.

Proof. (a) Choose any $g_1, g_2 \in G$, so that Hg_1, Hg_2 are arbitrary in G/H . Then

$$\begin{aligned} (Hg_1)(Hg_2) &= H(g_1 g_2) && \text{(this is the operation on } G/H) \\ &= H(g_2 g_1) && \text{(since } G \text{ is abelian)} \\ &= (Hg_2)(Hg_1), && \text{(this is the operation on } G/H), \end{aligned}$$

so G/H is abelian.

(b) Let e be the identity of G , and note

$$\begin{aligned} (Hg)^{\text{ord}(g)} &= H(g^{\text{ord}(g)}) && \text{(this is the operation on } G/H) \\ &= He && \text{(definition of order)}. \end{aligned}$$

Since He is the identity of G/H , we know from class ([Cla, Fact 6.4]) that $\text{ord}(Hg)$ is a divisor of $\text{ord}(g)$. □

References

[Cla] Our Group Theory Class, [Our Definition Sheet](#).