HW 6 Due: 4 March 2026

As always, your answer will be graded on the quality of presentation as well as the correct answer. To get a good score: write your answer neatly, use complete sentences, and *justify your work.*

Computations

1. Let $G = \mathbb{Z}_{20}$ and $H = \langle 15 \rangle$. Write down all elements of all (right) cosets of H in G.

   Solution. Since $H = \{0, 5, 10, 15\}$, we have

   - $H = H + 0 = \{0, 5, 10, 15\}$,
   - $H + 1 = \{1, 6, 11, 16\}$,
   - $H + 2 = \{2, 7, 12, 17\}$,
   - $H + 3 = \{3, 8, 13, 18\}$, and
   - $H + 4 = \{4, 9, 14, 19\}$.

   $\square$

2. Let $G = D_4$ and $H = \langle R \rangle$. Write down all elements of all (right) cosets of H in G.

   Solution. Since $H = \left\{I, R, R^2, R^3\right\}$, we have

   - $H = HI = \left\{I, R, R^2, R^3\right\}$ and
   - $HF = \left\{F, FR^3, FR^2, FR\right\}$.

   $\square$

3. Define an equivalence relation on D_5 by setting, for all $g, h \in D_5$:

   $$g \sim h \qquad \text{if and only if} \qquad \langle R \rangle g = \langle R \rangle h.$$

   For $g \in D_5$, write $[g]$ for the equivalence class of g using this relation. Write down all elements in $[F]$.

   Solution. Since $\langle R \rangle = \left\{I, R, R^2, R^3, ^2\right\}$, we have

   - $\langle R \rangle I = \langle R \rangle R = \langle R \rangle R^2 = \langle R \rangle R^3 = \langle R \rangle R^4 = \langle R \rangle$ and
   - $\langle R \rangle F = \langle R \rangle RF = \langle R \rangle R^2 F = \langle R \rangle R^3 F = \langle R \rangle R^4 F = \left\{F, RF, R^2F, R^3F, R^4F\right\}$.

   Thus, we see $[F] = \left\{R, RF, R^2F, R^3F, R^4F\right\}$. $\square$

4. Define the function

   $$\begin{aligned} \phi\colon \mathbb{Z}_{10} &\to \mathbb{Z}_{10} \\ n &\mapsto n + n. \end{aligned}$$

   Let $\sim$ be the equivalence relation on $\mathbb{Z}_{10}$ obtained as in (V). From class, we know that this equivalence relation gives a partition of $\mathbb{Z}_{10}$. Write down this partition.

Solution. $\{\{0, 5\}, \{1, 6\}, \{2, 7\}, \{3, 8\}, \{4, 9\}\}$. $\square$

Proofs

(I) Let G be a group and define a relation $\sim$ on G by the condition: for all $g_1, g_2 \in G$,

$$g_1 \sim g_2 \qquad \text{if and only if} \qquad \text{there exists } h \in G \text{ such that } g_1 = hg_2h^{-1}.$$

Prove that $\sim$ is an equivalence relation.

Proof. We must prove three things. Let's write e for the identity of G.

- To prove $\sim$ is reflexive, choose any g. Since $g = ege = ege^{-1}$, we see $g \sim g$.
- To prove $\sim$ is symmetric, choose any $g_1, g_2 \in G$ and assume that $g_1 \sim g_2$. By definition of $\sim$, this means that there exists $h \in G$ such that $g_1 = hg_2h^{-1}$. Well then, this means $g_2 = h^{-1}g_1h = h^{-1}g_1\left(h^{-1}\right)^{-1}$, so that $g_2 \sim g_1$.
- Finally, to prove $\sim$ is transitive, choose any $g_1, g_2, g_3 \in G$ and assume that $g_1 \sim g_2$ and $g_2 \sim g_3$. By definition of $\sim$, this means that there exist $h_1, h_2 \in G$ such that $g_1 = h_1g_2h_1^{-1}$ and $g_2 = h_2g_3h_2^{-1}$. But then, using associativity and shoes and socks:

$$g_1 = h_1g_2h_1^{-1} = h_1\left(h_2g_3h_2^{-1}\right)h_1^{-1} = (h_1h_2)\,g_3\left(h_2^{-1}h_1^{-1}\right) = (h_1h_2)\,g_3\,(h_1h_2)^{-1},$$

  so $g_1 \sim g_3$.

□

(II) Let G be a group, let H be a subgroup of G, and let $a, b \in G$. Prove

$$Ha = Hb \qquad \text{if and only if} \qquad ab^{-1} \in H.$$

Proof. Let e be the identity element of G.

- First, suppose $Ha = Hb$. Since $e \in H$, there exists $h \in H$ with $a = ea = hb$. Then $ab^{-1} = h \in H$.
- Conversely, suppose there is some $h_0 \in H$ with $ab^{-1} = h_0$. We prove $Ha = Hb$ in two steps:
  - To see that $Ha \subseteq Hb$, choose any $h \in H$ and note that $ha = h\,(h_0b) = (hh_0)\,b \in Hb$.
  - For the other inclusion, choose any $h \in H$ and note that $hb = h\left((h_0)^{-1}a\right) = \left(h\,(h_0)^{-1}\right)a \in Ha$.

□

(III) Suppose n is a positive odd integer, that G is a group of order $2n$, and that $a, b \in G$ have orders $n, 2$, respectively. Prove that

$$G = \left\{a^ib^j \mid i \in \{0, \ldots, n-1\} \text{ and } j \in \{0, 1\}\right\}.$$

Proof. Since $\operatorname{ord}(a) = n$, we know that $|\langle a\rangle| = n$, so that $[G : \langle a\rangle] = \frac{2n}{n} = 2$. That is, there are *precisely two* cosets of $\langle a\rangle$ in G. By Lagrange's Theorem, the fact that n is odd and $\operatorname{ord}(b) = 2$ tells us that $b \notin \langle a\rangle$, so the two cosets of $\langle a\rangle$ that partition G are:

- $\langle a\rangle = \left\{a^0, \ldots, a^{n-1}\right\}$ and
- $\langle a\rangle\, b = \left\{a^0b, \ldots, a^{n-1}b\right\}$.

□

(IV) Let $\mathcal{F}$ be the set of all functions with domain and codomain $\mathbb{R}$. Define $\sim$ on $\mathcal{F}$ by setting for all $f, g \in \mathcal{F}$:

$$f \sim g \qquad \text{if and only if} \qquad f(0) = g(0).$$

Prove that $\sim$ is an equivalence relation on $\mathcal{F}$.

Proof. We must prove three things.

- To prove $\sim$ is reflexive, choose any $f \in \mathcal{F}$. Since $f(0) = f(0)$, we see $f \sim f$.
- To prove $\sim$ is symmetric, choose any $f, g \in \mathcal{F}$ and assume that $f \sim g$. By definition of $\sim$, this means that $f(0) = g(0)$. Well then, this means $g(0) = f(0)$ too, so that $g \sim f$.
- Finally, to prove $\sim$ is transitive, choose any $f, g, h \in \mathcal{F}$ and assume that $f \sim g$ and $g \sim h$. By definition of $\sim$, this means that $f(0) = g(0)$ and $g(0) = h(0)$. But then, by transitivity of equality, we see that $f(0) = h(0)$, so that $f \sim h$.

□

(V) Suppose S, T are sets and $f: S \to T$ is a function. Define $\sim$ on S by the condition: for all $s_1, s_2 \in S$,

$$s_1 \sim s_2 \qquad \text{if and only if} \qquad f(s_1) = f(s_2).$$

Prove that $\sim$ is an equivalence relation on S.

Proof. We must prove three things.

- To prove $\sim$ is reflexive, choose any $s \in S$. Since $f(s) = f(s)$, we see $s \sim s$.
- To prove $\sim$ is symmetric, choose any $s_1, s_2 \in S$ and assume that $s_1 \sim s_2$. By definition of $\sim$, this means that $f(s_1) = f(s_2)$. Well then, this means $f(s_2) = f(s_1)$ too, so that $s_2 \sim s_1$.
- Finally, to prove $\sim$ is transitive, choose any $s_1, s_2, s_3 \in S$ and assume that $s_1 \sim s_2$ and $s_2 \sim s_3$. By definition of $\sim$, this means that $f(s_1) = f(s_2)$ and $f(s_2) = f(s_3)$. But then, by transitivity of equality, we see that $f(s_1) = f(s_3)$, so that $s_1 \sim s_3$.

□